

Vortex ordering in fully frustrated superconducting systems with a dice lattice

S. E. Korshunov

L. D. Landau Institute for Theoretical Physics, Kosygina 2, Moscow 117940, Russia

(Received 26 July 2000; published 28 February 2001)

The structure and the degeneracy of the ground state of a fully frustrated XY model are investigated for the case of a dice-lattice geometry. The results are applicable for the description of Josephson junction arrays and thin superconducting wire networks in the external magnetic field providing half-integer number of flux quanta per plaquette. The mechanisms of disordering of vortex pattern in such systems are briefly discussed.

DOI: 10.1103/PhysRevB.63.134503

PACS number(s): 74.80.-g, 75.10.Hk, 64.60.Cn

I. INTRODUCTION

De Gennes¹ and Alexander² have shown that the linearized Ginzburg-Landau equations for a superconducting wire network in external magnetic field can be mapped on the eigenvalue equations for a single electron hopping problem in the same geometry. Thus the dependence of a (mean-field) superconducting transition temperature on external field can be found by following the field dependence of the lowest eigenvalue in single electron problem.

Recently it has been shown by Vidal *et al.*³ that the single electron hopping problem has very special features in the case of so-called^{4,5} dice lattice (see Fig. 1) if the value of the magnetic flux per plaquette Φ is equal to one-half of the flux quantum $\Phi_0 = hc/2e$. Namely, the spectrum of an electron lacks any dispersion and is reduced to three discrete levels.

This theoretical result has led to the interest in experimental investigation of superconducting networks with dice-lattice geometry.^{6,7} It has been shown⁷ that in contrast to the case of $\Phi = \Phi_0/3$, for which vortex pattern in dice network is nicely ordered, at $\Phi = \Phi_0/2$ [so-called fully frustrated (FF) case] the vortices do not form (presumably at the same temperature) any regular pattern. The authors of Ref. 7 have suggested that this absence of ordering is related with an infinite degeneracy and localized structure of the states corresponding to the lowest energy level in terms of a single electron problem (or to the lowest free energy in terms of a superconducting network).

Although this conjecture can be correct, it still has to be verified. A superconducting network problem reduces to a single electron hopping problem only at the mean-field transition point. Below it the nonlinear terms in Ginzburg-Landau equation become important and may completely or partially remove the high degeneracy of the state with the lowest free energy.

In the present work we use a different approach for theoretical investigation of an ordering in a FF superconducting system with dice lattice. We consider another limit when the amplitude of the order parameter is well defined and uniform, but phase fluctuations are possible and can lead to destruction of an ordered state. In that limit a discrete superconducting system (a wire network or a junction array) in external magnetic field can be described by a frustrated XY model introduced in Sec. II.

In Sec. III we propose a highly symmetric state that due to simplicity of its structure may be a good candidate for the

ground state of a FF XY model with dice lattice. In Sec. IV we show that this state has high additional degeneracy because it allows for formation of zero-energy domain walls. Sec. V is devoted to a brief discussion of possible consequences of this additional degeneracy for the disordering of vortex pattern in FF superconducting systems with dice-lattice geometry.

II. THE MODEL

In the regime when only phase fluctuations are of importance an array of weakly coupled superconducting islands can be described by the Hamiltonian

$$H = \sum_{\langle ij \rangle} V(\theta_{ij}), \quad (1)$$

where the sum is performed over all pairs of coupled islands and

$$\theta_{ij} = \varphi_j - \varphi_i - \frac{2\pi}{\Phi_0} \int_i^j d\mathbf{x} \mathbf{A}(\mathbf{x}) \equiv -\theta_{ji} \quad (2)$$

is the gauge-invariant phase difference that can be associated with the link $\langle ij \rangle$. Here φ_j is the order parameter phase of j th superconducting island and \mathbf{A} is the vector potential. The phases φ_j are defined up to a shift by a multiple of 2π , therefore the interaction function $V(\theta)$ has to be periodic in θ . The form of $V(\theta)$ depends on the type of the coupling. For Josephson junction array

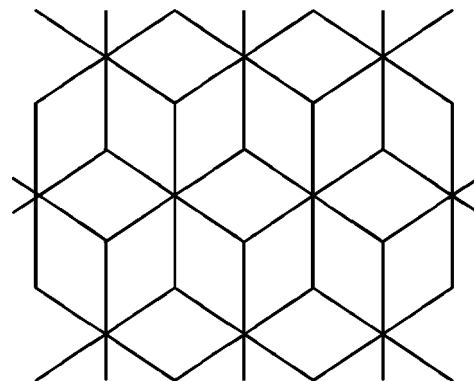


FIG. 1. Dice lattice is periodic and has hexagonal symmetry. It consists of the sites with coordination numbers 3 and 6. All elementary plaquettes are rhombic.

$$V(\theta) = -J \cos \theta, \quad (3)$$

where J is the coupling constant of the junction.

Summation of Eq. (2) over a perimeter of a lattice plaquette imposes a constraint

$$\sum_{\square} \theta_{ij} = -2\pi f, \quad (4)$$

where the frustration parameter f is equal to Φ/Φ_0 and Φ is the magnetic flux threading the plaquette. In the limit when screening effects can be neglected, Φ is determined by the external field and for the uniform field and flat geometry is proportional to the area of the plaquette. If all the plaquettes have equal areas the value of f is the same for all plaquettes. In such cases a system is called uniformly frustrated.

When interaction function $V(\theta)$ is periodic in θ , it is convenient to consider variables θ_{ij} reduced to the interval $[-\pi, \pi]$. That makes the description of any state in terms of θ_{ij} more transparent, but transforms the constraint (4) into

$$\sum_{\square} \theta_{ij} = 2\pi M, \quad M \equiv m - f, \quad (5)$$

where m is an integer. The form of Eq. (5) shows that f can be reduced to the interval $-1/2 < f \leq 1/2$, all other values of f being equivalent to some value from that interval. If $V(\theta)$ is an even function of θ (as it usually is) f is additionally equivalent to $-f$. The special case of $f = 1/2$ (which is the maximal irreducible value of f) is called a FF XY model.

Well below mean-field transition temperature a network of thin superconducting wires can be described by the same Hamiltonian (1) with the term $\varphi_j - \varphi_i$ in the definition of θ_{ij} [Eq. (2)] now substituted by the integral $\int_i^j dx (d\varphi/dx)$ along the link $\langle ij \rangle$.⁸ In that case summation of Eq. (2) around a perimeter of a plaquette leads directly to Eq. (5). In the limit of long thin wires the interaction function is almost harmonic:⁸

$$V(\theta) \propto \theta^2. \quad (6)$$

In this work we use the term ‘‘frustrated XY model’’ for a system defined by Eqs. (1) and (5) with a general form of the interaction function and not only for $V(\theta)$ of the form (3). Thus our approach is valid for the description both of junction arrays and wire networks.

III. THE GROUND STATE

In a FF XY model all variables M are half integer and different low-lying extrema of the Hamiltonian can be characterized by the distribution of positive and negative half vortices ($M = \pm 1/2$) in the plaquettes of the lattice. The vortices of the same sign repel each other, therefore in the ground state they can be expected to be situated as far from each other as possible. In particular, in the case of a FF XY model with square lattice the positive and negative half vortices form, in the ground state, a regular checkerboard pattern.⁹ Analogous pattern in which the nearest neighbors of each half vortex are of the opposite sign is possible when

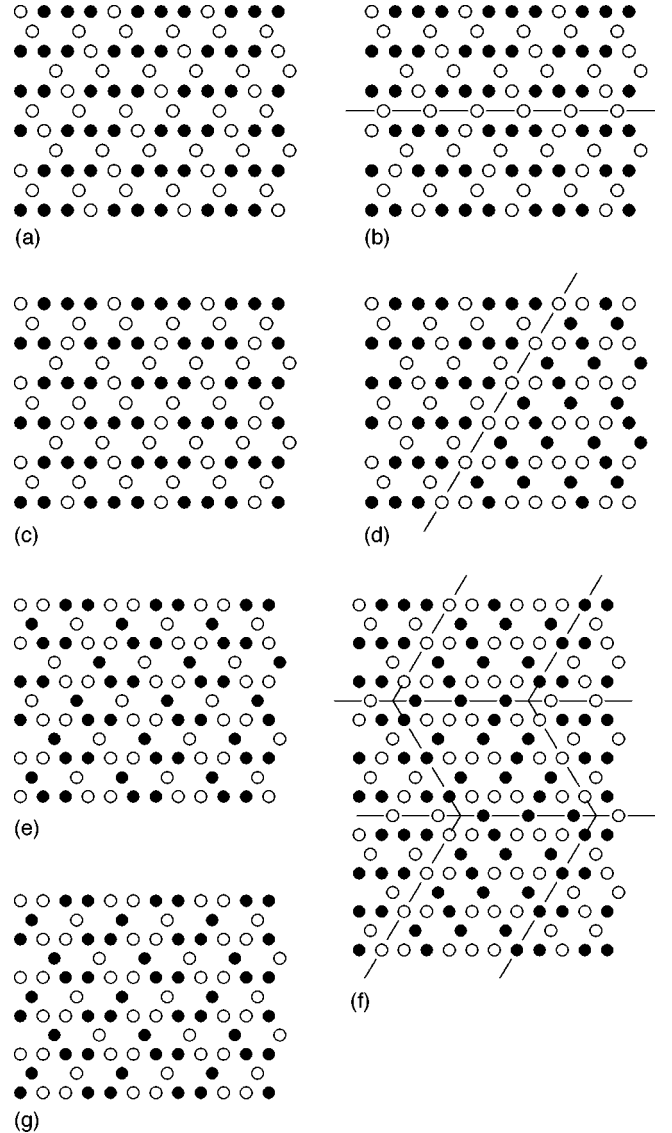


FIG. 2. Filled (empty) circles designate positive (negative) half vortices. Structures (a), (c), (e), and (g) are periodic, whereas (b), (d), and (f) include zero-energy domain walls separating different periodic states. All states shown have the same energy.

they occupy the sites of a honeycomb lattice, that is in FF XY model with triangular lattice.^{10,11}

A dice lattice is dual to a *kagomé* lattice, therefore in a FF XY model with dice lattice, the half vortices can be considered as occupying the sites of a *kagomé* lattice. Since a *kagomé* lattice is constructed from triangles it is impossible to distribute the half vortices in it in such a way that all nearest neighbors are of the opposite sign. The half vortices of the same sign will have to form clusters and the minimal size of such clusters that allow for covering of a *kagomé* lattice turns out to be equal to three. The most symmetric example of a regular (periodic) arrangement of vortices on a *kagomé* lattice in which half vortices of the same sign form the clusters of the size three (triads) is shown in Fig. 2(a).

This state has the 12-fold degeneracy and can be described as a regular lattice of vacancies (absent positive vortices) on the background of $f = 2/3$ ground state or, equiva-

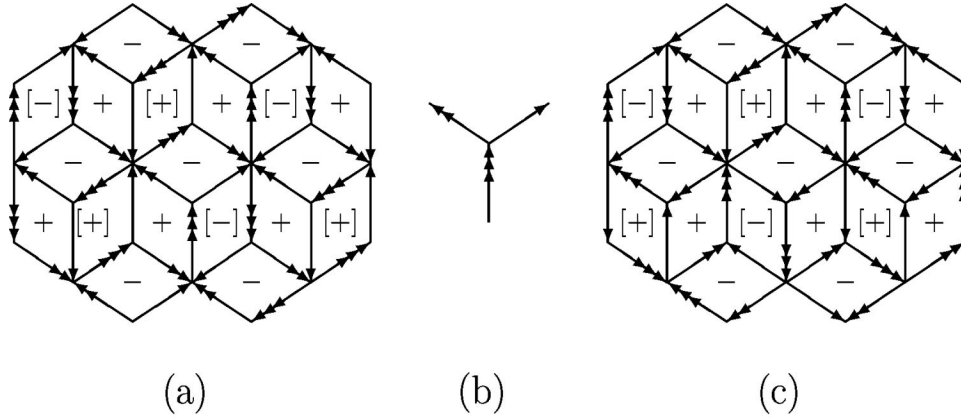


FIG. 3. (a) Phase representation of the periodic state shown in Fig. 2(a); (b) elementary pattern, repetition of which allows to construct this state; (c) phase representation of a zero-energy domain wall [shown in Fig. 2(b)]. Three types of arrows correspond to three different values of θ_{ij} .

lently, a regular lattice of extra positive vortices on the background of $f=1/3$ ground state (cf. with Ref. 7). However, symmetry considerations show that its structure in terms of gauge-invariant phase variables θ_{ij} [which is shown in Fig. 3(a)] is very simple and can be constructed by repetition (with rotation and reflection) of a simple three-link pattern shown in Fig. 3(b). Although we cannot rigorously prove that this state has the lowest possible energy (which is typical for XY models with nontrivial frustration), we believe that the simplicity of its structure strongly supports this conjecture.

In the state depicted in Fig. 3(a) the variables θ_{ij} acquire only three different values that we denote θ_a ($a=1,2,3$; $0 < \theta_1 < \theta_2 < \theta_3 < \pi$) and show as single, double, and triple arrows. The half vortices of the same sign are separated by single arrows, the central half vortex of each triad is separated from its neighbors of the opposite sign by triple arrows and the lateral half vortices of opposite sign are separated by double arrows. The same set of rules for extracting the distribution of θ_{ij} from the distribution of half vortices applies also to all the other states with the same energy discussed below.

The energy of the considered state (calculated per triple site of a dice lattice) is given by

$$E = V(\theta_1) + V(\theta_2) + V(\theta_3), \quad (7)$$

whereas general constraints (5) are reduced to

$$2\theta_1 + 2\theta_3 = \pi \quad (8)$$

for the central half vortex of each cluster and

$$-\theta_1 + 2\theta_2 + \theta_3 = \pi \quad (9)$$

for all the other (lateral) half vortices. Variation of Eq. (7) with constraints (8) and (9) gives

$$V'(\theta_1) + V'(\theta_2) = V'(\theta_3), \quad (10)$$

which (not unexpectedly) coincides with the condition of the current conservation for each of the triple sites. The current

conservation at each of the six-link sites is then ensured automatically.

For $V(\theta)$ of the form (3) (corresponding to Josephson junction array) the solution of Eqs. (8)–(10) gives

$$\theta_1 = \arctan\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \approx 10^\circ, \quad \theta_2 = \frac{\pi}{4} + \theta_1 \approx 55^\circ, \quad (11)$$

$$\theta_3 = \frac{\pi}{2} - \theta_1 \approx 80^\circ;$$

whereas for $V(\theta) \propto \theta^2$ (the case of a thin wire network)

$$\theta_1 = 15^\circ, \quad \theta_2 = 60^\circ, \quad \theta_3 = 75^\circ. \quad (12)$$

Thus the values of θ_a are only weakly sensitive to the type of superconducting system that manifests itself in the form of the current-phase relation.

IV. THE ADDITIONAL DEGENERACY

The regular state depicted in Fig. 3(a) [Fig. 2(a)] allows for the construction of zero-energy domain wall. Figure 3(c) shows how one can rearrange the arrows in the lower half of Fig. 3(a) without invalidating constraints (5) or current conservation relations, obtaining in such a way another extremum with the same energy. The same state is shown in Fig. 2(b) in terms of distribution of half vortices. It looks like a domain wall separating the state (a) [i.e., the state shown in Fig. 2(a)] from another version of the same state in which the triads of negative half vortices change their orientation by 60° .

It is possible to construct such domain wall on each horizontal line similar to the line shown in Fig. 2(b). This increases the degeneracy by the factor 2^N (where N is the number of available positions of the domain walls) due to the binary possibility of having or not having a domain wall at each available position. The regular state constructed by inserting into the state of Fig. 2(a) a domain wall at each available position is shown in Fig. 2(c). This state is also periodic, but has the higher (24-fold) degeneracy than the state (a).

Insofar we have discussed only the zero-energy domain walls that are parallel to the triads of positive half vortices. It follows from the symmetry considerations that analogous domain walls can also be constructed in parallel to the triads of negative half vortices. However, the energy remains the same only if all domain walls of this type have the same orientation, therefore the total increase of the degeneracy due to a possible creation of zero-energy domain walls of the type (b) is given by a factor 2^{N+1} . Analogous restriction for the creation of zero-energy domain walls appears in the case of the frustrated XY model with a triangular lattice and $f=1/4$ or $f=1/3$.¹²

In Fig. 3 the central half vortex of each triad is marked by square brackets. It is not hard to notice that all the other (lateral) half vortices form the rows of alternating pluses and minuses. These rows are straight for the regular state of Fig. 3(a) [Fig. 2(a)], but the presence of domain walls of the type (b) makes them bend (in parallel to each other).

It turns out possible to interchange pluses and minuses in any of these rows by interchanging the single and triple arrows on the links that separate the row from the neighboring central half vortices and reversing the double arrows on all the links inside the row. This procedure does not invalidate current conservation at any site and does not change the energy of the system.

Such sign reversal in the rows of alternating lateral half vortices allows to construct the zero-energy domain wall of different type shown in Fig. 2(d), which [like the domain wall of the type (b)] also separates two different versions of the state (a), but now with interchanged orientations of positive and negative triads. A regular repetition of such domain wall at each available position leads to the periodic state shown in Fig. 2(e), which like the state (a) has the 12-fold degeneracy.

The zero-energy domain walls of different types can cross each other [as is shown in Fig. 2(f)] without increasing the energy of the system. Each time the wall of type (d) crosses the wall of the type (b) it has to change its orientation by 60° , all the walls of the type (d) being parallel to each other in each strip between the walls of the type (b).

Thus the system with an arbitrary number of the walls of the type (b) can simultaneously contain an arbitrary number of the walls of type (d). The total additional degeneracy [in comparison with our reference state (a)] is given by the factor 2^{N+M+1} , where M is the number of positions available for domain walls of the type (d), which is the number of rows of the alternating lateral half vortices. The most dense network of zero-energy domain walls of both types produces the periodic state shown in Fig. 2(g), which like the state (c) is characterized by the 24-fold degeneracy.

V. DISCUSSION

The zero-energy domain walls of the two types described above produce the contribution to residual (zero-temperature) entropy that grows with the increase of the system as $(N+M)\ln 2$, that is slower than its area (which is

proportional to NM). That is, they do not lead to appearance of the extensive residual entropy in contrast to the case of the antiferromagnetic XY model with a *kagomé* lattice in which the manifold of the ground states allows for construction of zero-energy domain walls that can form independent closed loops of arbitrary length.^{13,14} This property of the antiferromagnetic XY model with *kagomé* lattice leads both to a finite extensive entropy^{13,14} and to the presence of a hierarchical sequence of barriers,¹⁴ which may explain the experimentally observed glasslike dynamics of the antiferromagnet with such structure.¹⁵ The family of the states considered in this work demonstrates less developed degeneracy (analogous to that encountered in $f=1/3$ XY model with triangular lattice¹²).

Since different values of θ_a correspond to different values of $V''(\theta)$, at finite temperatures, the accidental degeneracy related to possible formation of zero-energy domain walls can be expected to be removed due to the difference in free energy of the small amplitude continuous fluctuations (spin waves), in the same way as it happens in the XY model with triangular lattice and $f=1/4$ or $f=1/3$.¹² Most probably the spin wave contribution to free energy will be minimal for one of the periodic states shown in Fig. 2, which therefore will be dominant in the low temperature limit.

The zero-energy domain walls separating from each other the different versions of this state will then acquire a positive free energy, so the phase transition associated with their proliferation (and vortex-pattern disordering) can be expected to happen at finite temperature. However, in the thin wire networks with almost harmonic interaction function $V(\theta)$ the effects related to the differences in spin wave free energy will be extremely weak, and therefore, the disordering of vortex pattern due to proliferation of domain walls may happen already at rather low temperatures.

The strong disordering of vortex pattern observed in FF superconducting network with dice lattice geometry⁷ can also have some relation to geometrical irregularities. Gupta and Teitel¹⁶ have recently shown that in the case of the FF XY model with square lattice, the irregularities of so-called ‘‘positional-disorder’’-type (uncorrelated lattice sites displacements, etc.) produce an effective random field for the Ising-type variables M (describing the signs of the half vortices) and therefore induce the destruction of long-range order (at large enough scales) even if the disorder is small. In the system that allows for formation of the zero-energy domain walls, the relevance of this mechanism may be strongly amplified.

ACKNOWLEDGMENTS

This work was supported in part by the Program ‘‘Statistical Physics’’ of the Russian Ministry of Science, by the Swiss National Science Foundation, and by the Netherlands Organization for Scientific Research (NWO) in the framework of Russian-Dutch Cooperation Program. The author is grateful to P. Martinoli and B. Pannetier for introduction to the problem and discussion of the results.

- ¹P. G. de Gennes, C. R. Seances Acad. Sci., Ser. 2 **292**, 9 (1981); **292**, 279 (1981).
- ²S. Alexander, Phys. Rev. B **27**, 1541 (1983).
- ³J. Vidal, R. Mosseri, and B. Douçot, Phys. Rev. Lett. **81**, 5888 (1998).
- ⁴T. Horiguchi and C.C. Chen, J. Math. Phys. **15**, 659 (1974).
- ⁵B. Sutherland, Phys. Rev. B **34**, 5208 (1986).
- ⁶C.C. Abilio, P. Boutaud, Th. Fournier, B. Pannetier, J. Vidal, S. Tedesco, and B. Dalzotto, Phys. Rev. Lett. **83**, 5102 (1999).
- ⁷B. Pannetier, C.C. Abilio, E. Serret, Th. Fournier, P. Butaud, and J. Vidale, cond-mat/0005254 (unpublished).
- ⁸S. Alexander and E. Halevi, J. Phys. (Paris) **44**, 805 (1983).
- ⁹J. Villain, J. Phys. C **10**, 1717 (1977).
- ¹⁰S. Miyashita and J. Shiba, J. Phys. Soc. Jpn. **53**, 1145 (1984).
- ¹¹D.H. Lee, R.G. Caflisch, J.D. Joannopoulos, and F.Y. Wu, Phys. Rev. B **29**, 2680 (1984).
- ¹²S.E. Korshunov, A. Vallat, and H. Beck, Phys. Rev. B **51**, 3071 (1995).
- ¹³D.A. Huse and A.D. Rutenberg, Phys. Rev. B **45**, 7536 (1992).
- ¹⁴I. Ritchey, P. Chandra, and P. Coleman, Phys. Rev. B **47**, 15 342 (1993); P. Chandra, P. Coleman, and I. Ritchey, J. Phys. I **3**, 591 (1993).
- ¹⁵A.S. Wills, V. Dupuis, E. Vincent, J. Hammann, and R. Calemczuk, cond-mat/0001344 (unpublished).
- ¹⁶P. Gupta and S. Teitel, Phys. Rev. Lett. **82**, 5313 (1999).