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Analog of Kelvin–Helmholtz Instability on a Free Surface of a Superfluid Liquid[1]

S. E. Korshunov

Landau Institute for Theoretical Physics, Russian Academy of Sciences, ul. Kosygina 2, Moscow, 117940 Russia
e-mail: serkor@landau.ac.ru
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We analyze the analog of the Kelvin–Helmholtz instability on the free surface of a superfluid liquid. This instability is induced by the relative motion of superfluid and normal components of the same liquid along the surface. The instability threshold is found to be independent of the value of viscosity, but turns out to be lower than in the absence of dissipation. The result is similar to that obtained for the interface between two sliding superfluids (with different mechanisms of dissipation) and confirmed by the first experimental observation of the Kelvin–Helmholtz instability on the interface between ^{3}He-A and ^{3}He-B by Blaauwgeers *et al.* (cond-mat/0111343). *© 2002 MAIK "Nauka/Interperiodica".*

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1. INTRODUCTION

The Kelvin–Helmholtz instability [1] is a dynamic corrugation instability of the interface separating two liquids sliding with respect to each other. The concept of such instability was originally introduced when considering ideal liquids, and, in the presence of dissipation, it becomes ill-defined, because the relative motion of two liquids in contact with each other is no longer a solution to the hydrodynamic equations.

The simplest situation, where an equilibrium difference in velocities can be maintained at the surface of a liquid, is the relative motion of the superfluid and normal components (a counterflow) in superfluid ^{4}He. The corrugation instability of the free surface of a superfluid liquid in the presence of a counterflow along the surface was studied in [2] (in relation to the experiments of Egolf *et al.* [3]). It can be considered as an example of the Kelvin–Helmholtz instability, in which both liquids are located on the same side of the interface. An analogous instability can appear when superfluid ^{4}He slides along the atomically rough interface separating it from solid ^{4}He [4]. Such an interface is known to account for the equilibrium melting and crystallization of ^{4}He [5, 6], and, as a consequence, its behavior resembles that of the free surface of a liquid.

Recently, interest in surface instabilities of superfluids has been revived [7–9] in relation to the experiments on laser-manipulated Bose gases and the first experimental observation of the Kelvin–Helmholtz instability at the interface between two superfluids, ^{3}He-A and ^{3}He-B [10]. In particular, it has been demonstrated [9] that addition of a friction related to the motion of the interface with respect to container walls shifts the point of instability from the well-known classical threshold [1] to another value. This value does not depend on the strength of dissipation and can be reproduced in the framework of thermodynamic analysis by looking for the instability of free energy calculated in the reference frame of the normal component, which, in equilibrium, is at rest with respect to the container walls. The appearance of the same threshold in dynamic analysis was ascribed in [9] to the symmetry breaking related to the violation of the Galilean invariance by the considered friction force.

In this work, we return to the investigation of the corrugation instability on the free surface of a superfluid liquid in the presence of a counterflow [2] taking into account the viscosity of the normal component and show that, for any finite value of viscosity, the instability threshold is shifted to a viscosity-independent value, which is in agreement with the results of [9]. However, in our analysis, this phenomenon appears in the absence of the friction force violating the Galilean invariance. Therefore, the modification of the instability criterion in the presence of dissipation is not a consequence of the symmetry-breaking form of the friction, but has a more general nature.

2. DISPERSION RELATION

The calculation of the spectrum of surface oscillations in a superfluid liquid in the presence of a counterflow can be performed in the same way as the calculation of the spectrum of a gravitational wave in a normal liquid with finite viscosity [11]. For frequencies small in comparison with the frequency of the first and the second sound, the mass and the entropy densities can be

[1] This article was submitted by the author in English.

assumed to be constant. Accordingly, the conservation laws for mass and entropy are reduced to the constraints

$$\mathrm{div}\mathbf{v}_s = \mathrm{div}\mathbf{v}_n = 0, \qquad (1)$$

where $\mathbf{v}_s$ and $\mathbf{v}_n$ are the superfluid and normal velocities, respectively. In this limit, the Navier–Stokes equation for a superfluid liquid can be written as [12]

$$\rho_s\left[\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s\nabla)\mathbf{v}_s\right] + \rho_n\left[\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n\nabla)\mathbf{v}_n\right] = -\nabla p - \rho\mathbf{g} + \eta\Delta\mathbf{v}_n, \qquad (2)$$

where ρ_s and ρ_n are, respectively, superfluid and normal densities ($\rho = \rho_s + \rho_n$ being the total density); ρ is the pressure; $\mathbf{g}$ is the free fall acceleration; and η is the viscosity.

The solution to Eqs. (1, 2), satisfying the constraint $\mathrm{curl}\mathbf{v}_s = 0$ and corresponding to a small-amplitude surface wave with frequency ω and wavevector $\mathbf{q}$ parallel to the surface (we assume that, in equilibrium, the liquid is situated at $z < 0$) can be chosen in the form

$$\mathbf{v}_s^{\parallel}(\mathbf{r}, t) = \mathbf{v}_s^0 + i\mathbf{q}\gamma e^{qz}A, \qquad (3)$$

$$v_s^z(\mathbf{r}, t) = q\gamma e^{qz}A, \qquad (4)$$

$$\mathbf{v}_n^{\parallel}(\mathbf{r}, t) = \mathbf{v}_n^0 + i\mathbf{q}\gamma(e^{qz}B + e^{kz}C), \qquad (5)$$

$$v_n^z(\mathbf{r}, t) = \gamma[qe^{qz}B + (q^2/k)e^{kz}C], \qquad (6)$$

$$p(\mathbf{r}, t) = -\rho gz + i\gamma e^{qz}[\rho_s(\omega - \mathbf{v}_s^0\mathbf{q})A + \rho_n(\omega - \mathbf{v}_n^0\mathbf{q})B], \qquad (7)$$

where superscript $\parallel$ refers to the component of a vector parallel to the surface; $\gamma \equiv \exp i(\mathbf{qr} - \omega\mathbf{t})$;

$$k = \sqrt{q^2 - i\frac{\rho_n}{\eta}(\omega - \mathbf{v}_n^0\mathbf{q})}, \quad \mathrm{Re}k > 0; \qquad (8)$$

A, B, and C are (arbitrary) constants; and the possibility of an equilibrium counterflow (characterized by $\mathbf{v}_s^0 \neq \mathbf{v}_n^0$) is taken into account.

Substitution of Eqs. (3–7) into the boundary conditions describing the conservation of mass and entropy,

$$v_s^z - (\mathbf{v}_s^{\parallel}\nabla^{\parallel})\zeta = v_n^z - (\mathbf{v}_n^{\parallel}\nabla^{\parallel})\zeta = \frac{\partial\zeta}{\partial t}, \qquad (9)$$

and mechanical equilibrium,

$$\eta(\nabla^z\mathbf{v}_n^{\parallel} + \nabla^{\parallel}v_n^z) = 0, \qquad (10)$$

$$-p + 2\eta\nabla^z v_n^z = \sigma(\nabla^{\parallel})^2\zeta, \qquad (11)$$

at the surface (whose deviation from the plane $z = 0$ is denoted by ζ and surface tension by σ) shows that they are compatible with each other for

$$\rho_s(\omega - \mathbf{v}_s^0\mathbf{q})^2 + \rho_n\left(\omega - \mathbf{v}_n^0\mathbf{q} + i\frac{2\eta q^2}{\rho_n}\right)^2 + \frac{4\eta^2q^3k}{\rho_n} = \rho gq + \sigma q^3. \qquad (12)$$

The derivation of Eq. (12) does not require the assumption that viscosity is small, so it is applicable for an arbitrary value of viscosity.

3. INSTABILITY THRESHOLDS FOR ZERO AND FINITE VISCOSITY

For $\rho_s = 0$ and $\sigma = 0$, Eq. (12) is transformed to the dispersion relation of a gravitational wave on the free surface of a normal liquid [11], whereas, in the limit of $\eta = 0$, it is reduced to the equation

$$(\omega - \mathbf{vq})^2 = gq + \frac{\sigma}{\rho}q^3 - \frac{\rho_n\rho_s}{\rho^2}(\mathbf{wq})^2 \qquad (13)$$

describing the spectrum of surface waves in a superfluid with the counterflow [2] derived in the framework of the nondissipative two-fluid description. Here, $\mathbf{v} = (\rho_s\mathbf{v}_s^0 + \rho_n\mathbf{v}_n^0)/\rho$ is the mass velocity and $\mathbf{w} = \mathbf{v}_n^0 + \mathbf{v}_s^0$ is the relative velocity in the superfluid. The form of Eq. (13) shows that the roots with positive and negative imaginary parts (the former correspond to growing corrugation) exist only if the right-hand side can be negative, that is, if the absolute value of $\mathbf{w}$ exceeds w_{c0} defined by

$$w_{c0}^2 = \frac{2(\rho^3 g\sigma)^{1/2}}{\rho_n\rho_s}, \qquad (14)$$

with the instability taking place at $\mathbf{q} = \pm(\mathbf{w}/w)q_c$, where $q_c^2 = \rho g/\sigma$.

On the other hand, for any finite $\eta > 0$, one of the roots of Eq. (12) crosses the real axis already when

$$S(\mathbf{q}) \equiv gq + \frac{\sigma}{\rho}q^3 - \frac{\rho_s}{\rho}(\mathbf{wq})^2 \qquad (15)$$

touches zero, that is, at

$$|\mathbf{w}| = w_c \equiv \left[\frac{2(\rho g\sigma)^{1/2}}{\rho_s}\right]^{1/2} = \left(\frac{\rho_n}{\rho}\right)^{1/2}w_{c0}, \qquad (16)$$

with the instability appearing at the value of relative velocity *lower* than in the absence of dissipation, although at the same value of q. Note that, in the limit of zero temperature (when $\rho_s \longrightarrow \rho$), the criterion (16) coincides with the Landau criterion for the creation of ripplons in the reference frame of container walls.

For $S(\mathbf{q})$ sufficiently close to zero, the value of the root crossing the real axis is given by

$$\omega(\mathbf{q}) - \mathbf{v}_n^0\mathbf{q} \approx \frac{1}{2}\frac{\rho S(\mathbf{q})}{\rho_s \mathbf{w}\mathbf{q} + i\eta q^2}. \quad (17)$$

This shows that, for small viscosity and $\mathbf{w}$ just above w_c, the rate of instability development decreases with decreasing η, contrary to what is naturally expected.

By looking where the free energy of a corrugation, calculated in the reference frame of the normal component, is no longer positively defined (such an approach can be considered as a macroscopic generalization of the Landau criterion), the threshold for the instability of the interface between two different superfluids was found in [9] to be

$$\rho_{s1}(\mathbf{v}_{s1}^0 - \mathbf{v}_n^0)^2 + \rho_{s2}(\mathbf{v}_{s2}^0 - \mathbf{v}_n^0)^2 = 2(F\sigma)^{1/2}, \quad (18)$$

where F is a generalized restoring force, whose role in the case of a free surface is played by ρg. In the limit where the density of one of the liquids goes to zero, Eq. (18) is reduced to our criterion (16) obtained for the free surface of a single superfluid liquid.

4. CONCLUSION

In this work, we have investigated the dynamic instability of the free surface of a superfluid liquid caused by the relative motion of superfluid and normal components along the surface. The value of the instability threshold for finite viscosity, given by Eq. (16), is found to be independent of viscosity, but lower than in the absence of dissipation. The same criterion can be obtained by looking for the thermodynamic instability in the reference frame of the normal component.

An analogous modification of the instability threshold was found [9] to take place at the interface between two superfluids in the presence of friction with respect to the reference frame related to container walls,[2] which leads to violation of the Galilean invariance. Note that in our problem the same phenomenon appears in the situation where the form of dissipation does not imply the explicit selection of a particular reference frame. Nonetheless, the presence of dissipation (a finite value of viscosity) turns out to be sufficient to produce the same criterion for surface instability as in the case where the form of friction leads to the direct violation of the Galilean invariance.

The first experimental observation of the Kelvin–Helmholtz instability at the interface between ^{3}He-A and ^{3}He-B by Blaauwgeers *et al.* [10] unambiguosly demonstrated that it does indeed take place not for the classical, but for the modified value of the threshold. According to our results, the same can be expected from the instability on the free surface of superfluid ^{4}He.

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[2] The same type of dissipation was taken into account by Kagan [4] when studying the instability of the quantum interface between superfluid and solid ^{4}He.

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