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Analog of Kelvin–Helmholtz instability on superfluid liquid free surface

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We analyse the analog of the Kelvin–Helmholtz instability on free surface of a superfluid liquid. This instability is induced by the relative motion of superfluid and normal components of the same liquid along the surface. The instabilty threshold is found to be independent of the value of viscosity, but turns out to be lower than in absence of dissipation. The result is similar to that obtained for the interface between two sliding superfluids (with different mechanism of dissipation) and confirmed by the first experimental observation of the Kelvin-Helmholtz instability on the interface between ^{3}He-A and ^{3}He-B by Blaauwgeers et al. (cond-mat/0111343).

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1. Introduction. The Kelvin-Helmholtz instability [1] is a dynamic corrugation instability of the interface separating two liquids sliding with respect to each other. The concept of such instabilty has been orginally introduced when considering ideal liquids and in presence of dissipation becomes ill defined, because a relative motion of two liquids in contact with each other is no longer a solution of the hydrodynamic equations.

The simplest situation, when an equilibrium difference in velocities can be maintained at a surface of a liquid, is the relative motion of superfluid and normal components (a counterflow) in superfluid ^{4}He. The corrugation instability of a superfluid liquid free surface in presence of a counterflow along the surface was studied in Ref. [2] (in relation with experiments of Egolf et al. [3]). It can be considered as an example of the Kelvin-Helmholtz instability, in which both liquids are located on the same side of the interface. Analogous instability can appear when superfluid ^{4}He is sliding along the atomically-rough interface separating it from solid ^{4}He [4]. Such interface is known to allow for equilibrium melting and crystallization of ^{4}He [5, 6], and, as a consequence, its behavior resembles that of free surface of a liquid.

Recently the interest to surface instabilities in superfliuds has been revived [7 – 9] in relation with experiments on laser manipulated Bose gases and the first experimental observation of the Kelvin-Helmholtz instability on the interface between two superfluids, ^{3}He-A and ^{3}He-B [10]. In particular, it has been demonstrated [9] that addition of a friction related to the motion of the interface with respect to container walls shifts the point of instability from the well known classical threshold [1] to another value. This value does not depend on the strength of dissipation and can be reproduced in the framework of the thermodynamic analysis by looking for the instability of the free energy calculated in the reference frame of normal component, which in equilibrium is at rest with respect to container walls. The appearance of the same threshold in dynamic analysis has been ascribed in Ref. [9] to the symmetry breaking related with the violation of the Galilean invariance by the considered friction force.

In the present work we return to investigation of the corrugation instability on free surface of a supefluid liquid in presence of a counterflow [2], taking into account the viscosity of the normal component, and show that for any finite value of viscosity the instability threshold is shifted to viscosity independent value, which is in agreement with the results of Ref. [9]. However, in our analysis this phenomenon appears in absence of the friction force violating the Galilean invariance.Therefore, the modification of the instability criterion in presence of dissipation is not a consequence of the symmetry breaking form of the friction, but has a more general nature.

2. Dispersion relation. The calculation of the surface oscillations spectrum in superfluid liquid in presence of a counterflow can be performed in the same way as the calculation of the spectrum of gravitational wave in normal liquid with finite viscosity [11]. For frequencies small in comparison with that of the first and the second sound, the mass and the entropy densities can be assumed to be constant. Accordingly, the conservation laws for mass and entropy are reduced to constraints:

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$$\mathrm{div}\,\mathbf{v}_s = \mathrm{div}\,\mathbf{v}_n = 0\,, \qquad (1)$$

where $\mathbf{v}_s$ and $\mathbf{v}_n$ are superfluid and normal velocities. In that limit the Navier-Stokes equation for superfluid liquid can be written as [12]

$$\rho_s \left[\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s\nabla)\mathbf{v}_s\right] + \rho_n \left[\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n\nabla)\mathbf{v}_n\right] = = -\nabla p - \rho\mathbf{g} + \eta\Delta\mathbf{v}_n, \qquad (2)$$

where ρ_s and ρ_n are superfluid and normal densities ($\rho = \rho_s + \rho_n$ being the total density), p is the pressure, $\mathbf{g}$ is the free fall acceleration, and η is the viscosity.

Solution of Eqs.(1), (2), satisfying the constraint $\mathrm{rot}\,\mathbf{v}_s = 0$ and corresponding to a small amplitude surface wave with frequency ω and wavevector $\mathbf{q}$ parallel to the surface (we assume that in the equilibrium the liquid is situated at $z < 0$), can be chosen in the form

$$\mathbf{v}_s^{\|}(\mathbf{r},t) = \mathbf{v}_s^0 + i\mathbf{q}\gamma e^{qz}A, \qquad (3)$$

$$v_s^z(\mathbf{r},t) = q\gamma e^{qz}A, \qquad (4)$$

$$\mathbf{v}_n^{\|}(\mathbf{r},t) = \mathbf{v}_n^0 + i\mathbf{q}\gamma(e^{qz}B + e^{kz}C), \qquad (5)$$

$$v_n^z(\mathbf{r},t) = \gamma[qe^{qz}B + (q^2/k)e^{kz}C], \qquad (6)$$

$$p(\mathbf{r},t) = -\rho g z + + i\gamma e^{qz}[\rho_s(\omega - \mathbf{v}_s^0\mathbf{q})A + \rho_n(\omega - \mathbf{v}_n^0\mathbf{q})B], \qquad (7)$$

where superscript $\|$ refers to the component of a vector parallel to the surface, $\gamma \equiv \exp i(\mathbf{qr} - \omega\mathbf{t})$,

$$k = \sqrt{q^2 - i\frac{\rho_n}{\eta}(\omega - \mathbf{v}_n^0\mathbf{q})}\,, \quad \mathrm{Re}\,k > 0\,, \qquad (8)$$

A, B and C are (yet arbitrary) constants, and the possibility of an equilibrium counterflow (characterized by $\mathbf{v}_s^0 \neq \mathbf{v}_n^0$) is taken into account.

Substitution of Eqs. (3)–(7) into the boundary conditions describing the conservation of mass and entropy

$$v_s^z - (\mathbf{v}_s^{\|}\nabla^{\|})\zeta = v_n^z - (\mathbf{v}_n^{\|}\nabla^{\|})\zeta = \frac{\partial\zeta}{\partial t}\,, \qquad (9)$$

and mechanical equilibrium

$$\eta\left(\nabla^z\mathbf{v}_n^{\|} + \nabla^{\|}v_n^z\right) = 0, \qquad (10)$$

$$-p + 2\eta\nabla^z v_n^z = \sigma(\nabla^{\|})^2\zeta \qquad (11)$$

at the surface (whose deviation from the plane $z = 0$ is denoted by ζ and surface tension by σ), shows that they are compatible with each other for

$$\rho_s\left(\omega - \mathbf{v}_s^0\mathbf{q}\right)^2 + \rho_n\left(\omega - \mathbf{v}_n^0\mathbf{q} + i\frac{2\eta q^2}{\rho_n}\right)^2 + + \frac{4\eta^2q^3k}{\rho_n} = \rho g q + \sigma q^3. \qquad (12)$$

The derivation of Eq. (12) have not required to assume the viscosity small, so it is applicable for arbitrary value of viscosity.

3. Instability thresholds for zero and finite viscosity. For $\rho_s = 0$ and $\sigma = 0$ Eq. (12) is transformed into the dispersion relation of gravitational wave on free surface of normal liquid [11], whereas in the limit of $\eta = 0$ it is reduced to equation

$$(\omega - \mathbf{vq})^2 = gq + \frac{\sigma}{\rho}q^3 - \frac{\rho_n\rho_s}{\rho^2}(\mathbf{wq})^2 \qquad (13)$$

describing the spectrum of surface wave in superfliud with a counterflow [2] derived in the framework of the non-dissipative two-fluid description. Here $\mathbf{v} = (\rho_s\mathbf{v}_s^0 + \rho_n\mathbf{v}_n^0)/\rho$ is the mass velocity and $\mathbf{w} = \mathbf{v}_n^0 - \mathbf{v}_s^0$ the relative velocity in the superfluid. The form of Eq. (13) shows that the roots with positive and negative imaginary parts (the former corresponds to growing corrugation) exist only when the right-hand side can be negative, that is when the absolute value of $\mathbf{w}$ exceeds w_{c0} defined by

$$w_{c0}^2 = \frac{2(\rho^3 g\sigma)^{1/2}}{\rho_n\rho_s}, \qquad (14)$$

the instability taking place at $\mathbf{q} = \pm(\mathbf{w}/w)q_c$, where $q_c^2 = \rho g/\sigma$.

On the other hand, for any finite $\eta > 0$ one of the roots of Eq. (12) crosses the real axis already when

$$S(\mathbf{q}) \equiv gq + \frac{\sigma}{\rho}q^3 - \frac{\rho_s}{\rho}(\mathbf{wq})^2 \qquad (15)$$

touches zero, that is at

$$|\mathbf{w}| = w_c \equiv \left[\frac{2(\rho g\sigma)^{1/2}}{\rho_s}\right]^{1/2} = \left(\frac{\rho_n}{\rho}\right)^{1/2} w_{c0}, \qquad (16)$$

the instability appearing at the *lower* value of relative velocity than in absence of dissipation, although at the same value of q. Note that in the limit of zero temperature (when $\rho_s \to \rho$) the criterion (16) coinsides with the Landau criterion for creation of ripplons in the reference frame of container walls.

For $S(\mathbf{q})$ sufficiently close to zero the value of the root which crosses real axis is given by

$$\omega(\mathbf{q}) - \mathbf{v}_n^0\mathbf{q} \approx \frac{1}{2}\frac{\rho S(\mathbf{q})}{\rho_s\mathbf{wq} + i\eta q^2}. \qquad (17)$$

This shows that for small viscosity and w just above w_c the rate of the development of the instability decreases with decreasing η contrary to what is natural to expect.

By looking when the free energy of a corrugation, calculated in the reference frame of the normal component, is no longer positively defined (such approach can be considered as a macroscopic generalization of the Landau criterion), the threshold for the instability of the interface between two different superfluids has been found [9] to be given by

$$\rho_{s1}(\mathbf{v}_{s1}^0 - \mathbf{v}_n^0)^2 + \rho_{s2}(\mathbf{v}_{s2}^0 - \mathbf{v}_n^0)^2 = 2(F\sigma)^{1/2}, \quad (18)$$

where F is a generalized restoring force, the role of which in the case of free surface is played by ρg. In the limit when the density of one of the liquids goes to zero, Eq. (18) is reduced to our criterion (16) obtained for a free surface of a single superfluid liquid.

4. Conclusion. In the present work we have investigated the dynamic instability of a superfluid liquid free surface caused by the relative motion of superfluid and normal components along the surface. The value of the instabilty threshold for finite viscosity, given by Eq. (16), has been found to be independent of viscosity, but lower than in absence of dissipation. The same criterion can be obtained by looking for the thermodynamic instability in the reference frame of the normal component.

Analogous modification of the instability threshold has been found [9] to take place on the interface between two superfluids in presence of a friction with respect to the reference frame related with container walls[2)], which leads to violation of the Galilean invariance. Note that in our problem the same phenomenon appears in situation when the form of dissipation does not imply the explicit selection of the particular reference frame. Nonetheless, the presence of dissipation (a finite value of viscosity) turnes out to be sufficient to produce the same criterion for surface instability as in the case when the form of the friction leads to the direct violation of the Galilean invariance.

The first experimental observation of the Kelvin-Helmholtz instability on the interface between ^{3}He-A and ^{3}He-B by Blaauwgeers et al. [10] have unambigously demonstrated that it indeed takes place not at the classical, but at the modified value of the threshold. Acording to our results, the same can be expected from the instability on free surface of superfluid ^{4}He.

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