

Optimal fluctuation approach to a directed polymer in a random medium

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A modification of the optimal fluctuation approach is applied to study the tails of the free-energy distribution function $P_L(F)$ for an elastic string in quenched disorder both in the regions of the universal behavior of $P_L(F)$ and in the regions of large fluctuations, where the behavior of $P_L(F)$ is nonuniversal. The difference between the two regimes is shown to consist in whether it is necessary or not to take into account the renormalization of parameters by the fluctuations of disorder in the vicinity of the optimal fluctuation.

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A large variety of physical systems can be described in terms of an elastic string interacting with a quenched random potential. The role of such a string can be played by a domain wall in a two-dimensional magnet, a vortex line in a superconductor, a dislocation in a crystal, and so on; however, most often this class of systems is discussed under the name of a directed polymer in a random medium. The unfading interest in this problem is additionally supported by its resemblance to more complex systems with quenched disorder (e.g., spin glasses), as well as by its relation to the dynamics of a randomly stirred fluid and to stochastic growth (see Refs. 1 and 2 for reviews).

A number of properties of a random directed polymer can be easily deduced as soon as one knows $P_L(F)$, the free-energy distribution function for large polymer length L . In 1987 Kardar³ proposed a method for calculating the moments of $P_L(F)$ in the one-dimensional (1D) problem with δ -correlated potential. However, later it was understood⁴ that the approach of Ref. 3 allows one to find⁵ only the tail of $P_L(F)$ at large negative F (the left tail), so the conclusions on the width of this distribution function have to rely on the assumption that at large L it acquires a universal form

$$P_L(F) = P_*(F/F_*)/F_*, \quad (1)$$

incorporating the dependence on all parameters through a single characteristic free-energy scale $F_*(L) \propto L^\omega$, which therefore can be extracted from the form of the tail. Here and below it is assumed that F is counted off from its average.

In the present Rapid Communication we report the results of the first systematic investigation of $P_L(F)$ in the directed polymer problem, which makes a distinction between the interval of the universal behavior and the regions of large fluctuations, where $P_L(F)$ can be expected to deviate from Eq. (1). We demonstrate that the far-left tail of $P_L(F)$ is determined by the optimal fluctuation and apply an analogous approach for the analysis of the far-right tail.⁶ Inside the universal region, we rely on the generalization of this method accounting for the renormalization of different parameters by fluctuations.

This allows us to show that in the 1D case the exponent η_+ in the dependence $\ln P_L(F) \propto -|F|^{\eta_+}$ is changed from 3 in the universal part of the right tail to 5/2 outside of it (at the largest F). On the other hand, we find that in the left tail the behavior of $P_L(F)$ is qualitatively the same (with $\eta_- = 3/2$) in

both regions and that the analysis of Ref. 5 reproduces a correct estimate for $F_*(L)$ only as a result of this coincidence, because in reality it describes the *nonuniversal* part of the left tail. We also analyze how the tails of $P_L(F)$ are modified when the distribution of disorder is characterized by a finite correlation length.

The model. We consider an elastic string confined to a plane and interacting with a random potential $V(t, x)$ with zero mean and Gaussian statistics described by

$$\overline{V(t, x)V(t', x')} = \delta(t - t')U(x - x').$$

An overbar denotes the average with respect to disorder, and t is the coordinate along the string. Such a string can be described by the Hamiltonian

$$H = \int_0^t dt' \left\{ \frac{J}{2} \left[\frac{dx(t')}{dt'} \right]^2 + V[t', x(t')] \right\}. \quad (2)$$

The partition function of a string that starts at $t=0$ and ends at the point (t, x) is denoted $z(t, x)$ and its free energy $f(t, x) = -T \ln z(t, x)$, T being the temperature. The form of $z(t, x)$ depends on the initial condition at $t=0$. In particular, fixed initial condition, $x(t=0)=0$, corresponds to $z(0, x) = \delta(x)$, whereas free initial condition to $z(0, x) = 1$.

Derivation of $f(t, x)$ with respect to t shows⁸ that the evolution of $f(t, x)$ with the increase in t is governed by the Kardar-Parisi-Zhang (KPZ) equation⁹ (with the inverted sign of the nonlinear term)

$$f_t + (1/2J)f_x^2 - \nu f_{xx} = V(t, x), \quad (3)$$

where t plays the role of time, subscripts t and x denote partial derivatives, and $\nu = T/2J$. On the other hand, derivation of Eq. (3) with respect to x leads⁸ to the Burgers equation¹⁰ with random force,

$$u_t + uu_x - \nu u_{xx} = V_x(t, x)/J, \quad (4)$$

in which $u(t, x) = f_x(t, x)/J$ plays the role of velocity.

The replica approach to the directed polymer problem is based on calculating the moments $Z_n \equiv Z^n$ of the distribution of the partition function, $Z \equiv z(L, 0)$, and does not involve the analytical continuation of n to 0. Kardar³ has noticed that for any integer $n \geq 1$ (and large enough length L), Z_n with an exponential accuracy can be approximated by $\exp[-E_0(n)L/T]$, where $E_0(n)$ is the ground-state energy of n

bosons whose mass is equal to J/T^2 and interaction to $-U(x)/T$. On the other hand, since $Z^n \equiv \exp(-nF/T)$, Z_n can also be expressed in terms of $P_L(F)$, the distribution function of the free energy $F \equiv f(L, 0) = -T \ln Z$:

$$Z_n = \int_{-\infty}^{+\infty} dF P_L(F) \exp(-nF/T). \quad (5)$$

In the case of a δ -correlated random potential $U(x) = U_0 \delta(x)$, $E_0(n)$ can be found exactly.¹¹ This gives $Z_n \propto \exp[(JU_0^2/24T^5)n^3L]$, where the linear-in- n term in $E_0(n)$ has been omitted, because it can be eliminated by a constant shift of the potential $V(t, x)$ in Eq. (2). Comparison of the two expressions for Z_n allows one to conclude⁵ that for large negative fluctuations $P_L(F) \propto \exp[-S(F)]$, where

$$S(F) \approx \frac{2}{3} \left(\frac{-F}{F_*} \right)^{3/2}, \quad F_*(L) = \frac{1}{2} \left(\frac{JU_0^2 L}{T^2} \right)^{1/3}. \quad (6)$$

For such $P_L(F)$ the integral in Eq. (5) at positive n is dominated by the saddle point at $F \approx -F_c(L)n^2$, where

$$F_c(L) = F_*^3(L)/T^2 = JU_0^2 L / 8T^4 \gg F_*(L). \quad (7)$$

Since the explicit expression for Z_n is known only at integer $n \geq 1$ and cannot be analytically continued to $0 < n < 1$,¹⁴ the region of the applicability of Eq. (6) is restricted to $-F \gg F_c(L)$. With increase in L this region is shifted to larger and larger values of the ratio $|F|/F_* \gg F_c/L_* \propto L^{2/3}$, which means that the replica approach cannot provide any information on $P_*[F/F_*(L)]$ — i.e., on the universal form of $P_L(F)$ at $L \rightarrow \infty$.

The optimal fluctuation approach. If $V(t, x)$ were independent of t , for large enough L the free energy of a polymer in a given disorder realization would be given by $F \approx E_0 L$, where E_0 is the ground-state energy of the single-particle quantum-mechanical Hamiltonian, $\hat{H} = -(T^2/2J)\partial_x^2 + V(x)$. When V is fluctuating with t , one can nonetheless consider a question about the form of the most probable “stationary” (i.e., uniform along t) fluctuation of V which leads to the given negative value of E_0 . According to the analysis of Ref. 12, this form is given by

$$V(x) = -\Lambda \int_{-\infty}^{+\infty} dx' U(x-x') \Psi^2(x'), \quad (8)$$

where Λ is an arbitrary constant and $\Psi(x)$ is the localized solution of the nonlinear Schrödinger equation $E_0 \Psi = \hat{H} \Psi$, in which $V(x)$ has to be replaced by the right-hand side of Eq. (8).

In the case of a δ -correlated random potential $U(x) = U_0 \delta(x)$, such a solution (the soliton) has a form¹² $\Psi_\Delta(x) = T/[(\Lambda J U_0)^{1/2} \Delta \cosh(x/\Delta)]$, where the parameter $\Delta(F) = T[L/2J(-F)]^{1/2}$ can be called the soliton width. Substitution of $\Psi_\Delta(x)$ into Eq. (8) then gives

$$V(x) = -(T^2/J\Delta^2) \cosh^{-2}(x/\Delta). \quad (9)$$

In such a potential there exists only one level with a negative energy,¹³ from where the condition for the applicability of

the relation $F \approx E_0 L$ is $-F \gg T$. Note that both $\Delta(F)$ and $V(x)$ do not depend on Λ .

Substitution of Eq. (9) into the functional

$$S = \frac{1}{2U_0} \int_0^L dt \int_{-\infty}^{+\infty} dx V^2(t, x), \quad (10)$$

determining the probability of $V(t, x)$ for $U(x) = U_0 \delta(x)$, leads to exactly the same expression for $S(F)$ as follows from the replica analysis; see Eq. (6). This proves that for a δ -correlated random potential the far-left tail of $P_L(F)$ is entirely determined by the optimal fluctuation of $V(t, x)$. However, in order to find the range of the applicability of such an approach one has to consider the influence of the fluctuations of disorder in the vicinity of the optimal fluctuation. This can be done most transparently in terms of the Burgers equation representation.

The Burgers equation approach. It follows from the definition of $u(t, x) = f_x(t, x)/J$ that the stationary solution of the Burgers equation (4) with the pumping potential $V(x)$ of the form (9) is given by

$$u_\Delta(x) = -\frac{T}{J} \frac{d}{dx} \ln \Psi_\Delta(x) = u_\Delta \tanh(x/\Delta), \quad (11)$$

where $u_\Delta = T/J\Delta$. Thus, in terms of the velocity $u(t, x)$ the stationary soliton looks like an inverted standing shock wave.¹⁴ Such solitons have been discussed in a number of works by Fogedby.¹⁵

The description in terms of the Burgers equation allows one to understand rather easily how the shape of the optimal fluctuation evolves with time when one takes into account the initial condition. Below we adopt the free initial condition for the string, $z(0, x) = 1$, which in terms of interface dynamics corresponds to the standard initial condition for nonstationary growth, $u(0, x) = 0$.

If one assumes that for $t > 0$ the potential $V(x)$ is of the form (9), then the solution of Eq. (4) has to be close to Eq. (11) for not too large x , but must be close to zero at large enough distance from the region around $x=0$ where the pumping is localized. It is clear that this can be realized via the presence of two traveling shock waves, which does not require any additional pumping and therefore does not change $S(F)$. The velocities of these waves are determined by their amplitude, $v = \pm u_\Delta/2$, so for $t \gg \Delta/u_\Delta$ they will be located at $|x| \approx u_\Delta t/2 \gg \Delta$ — i.e., relatively far from the soliton core. Accordingly, in this regime $F \equiv J \int_{-\infty}^0 dx u(x)$ has to be close to $-(J/2)u_\Delta^2 t$, as has been assumed above. It is possible to check that the imposition of the fixed initial condition [$x(t=0)=0$] also does not lead to the change of $S(F)$.

In terms of the Burgers equation parameters (the viscosity $\nu = T/2J$ and the pumping force intensity $D = U_0/2J^2$), Eq. (6) can be rewritten as

$$S(F) = \frac{4}{3} \frac{\nu}{D} \left[\frac{2(-F)^3}{J^3 L} \right]^{1/2}. \quad (12)$$

From the nature of the optimal fluctuation approach it is clear that this expression can be expected to be valid only when the soliton is so narrow, $\Delta \ll x_0$ (where

$x_0 \sim \nu^3/D \sim T^3/JU_0$), that one can neglect the renormalization of parameters by the nonlinearity. Note that the constraint $\Delta \ll x_0$ is equivalent to the condition $-F \gg F_c(L)$ defining the region of the applicability of the replica approach results.

At $-F \ll F_c(L)$ (i.e., $\Delta \gg x_0$) one has to take into account the influence of the fluctuations in the vicinity of optimal fluctuation. Since optimal fluctuation is quasistationary, this (up to a numerical factor) can be done by replacing all parameters in Eq. (12) by their effective values at the scales of the order of Δ (and zero frequency). However, it is well known that the amplitude of the nonlinear term in Eq. (3) (and, therefore, coefficient J) cannot be renormalized¹⁶ due to the Galilean invariance. On the other hand, in the 1D case $\mu = D/\nu$ has to remain unchanged⁹ as a consequence of the fluctuation-dissipation theorem obeyed by Eq. (4).⁸ The coefficient μ describes the spectral density of the equilibrium equal-time fluctuations of u , which is known to be exactly the same as in the absence of the nonlinearity.

Since Eq. (12) includes only invariant coefficients $\mu = D/\nu$ and J , it can be expected to remain applicable not only for $-F \gg F_c(L)$, but also (with some numerical correction) in the region of the universal behavior, $-F \leq F_c(L)$. From the other side the region of the applicability of this estimate is restricted by the constraint $L \gg (\Delta^3/\mu)^{1/2}$, which is required¹⁷ for the attainment of equilibrium at the length scales of the order of Δ and is satisfied as soon as $-F \gg F_*(L)$ [i.e., $S(F) \gg 1$]. The fulfillment of this condition is necessary for the applicability of the above argument (based on known properties of an *equilibrated* system) in our nonstationary situation.

The right tail. The very special shape [in terms of $f(t, x)$] of the optimal fluctuation corresponding to the left tail originates from the possibility of having a growing fluctuation inside of which almost everywhere $f_t + (1/2J)f_x^2 \approx 0$, and therefore only a small part of the volume of this fluctuation contributes to the functional (10) [where $V(t, x)$ should be replaced by the left-hand side of Eq. (3)]. It is clear that in the case of the right tail such a situation is impossible because f_t has to have the same sign as f_x^2 . As a consequence, the optimal fluctuation corresponding to the right tail must have a more simple shape which can be characterized by a single relevant length scale Δ . This length scale can be estimated from $f_t \sim (1/2J)f_x^2$, which gives $\Delta(F) \sim (FL/J)^{1/2}$ and a temperature-independent estimate for $S(F)$,

$$S(F) \sim \frac{F^{5/2}}{U_0 J^{1/2} L^{1/2}}. \quad (13)$$

Note that Eq. (13) contains the unrenormalized coefficient U_0 , whereas the estimate for $\Delta(F)$ shows that it grows with increase in F . This suggests that even for large F it may be necessary to correct Eq. (13) by taking into account the renormalization of U_0 .

It follows from the analysis of higher-order diagrams¹⁸ that in a stationary 1D system the scale dependence of ν and $D \equiv U_0/2J^2$ can be found from the self-consistent one-loop theory (mode coupling approximation),^{19,20} which for

$\Delta \gg x_0$ gives $\nu(\Delta) \sim (\mu\Delta)^{1/2}$ and $D(\Delta) \sim (\mu^3\Delta)^{1/2}$.^{20,17} However, in the case of the right tail the replacement of D by $D(\Delta)$ (where Δ is the optimal fluctuation width) overestimates the renormalization of D . It is so because the only source for the renormalization of ν and D is the nonlinear term in Eq. (4), and therefore this renormalization can be effective only up to a length scale $R \sim \mu JL/F$ at which the typical velocity of equilibrium fluctuations, $u_R \sim (\mu/R)^{1/2}$, becomes comparable with the velocity $u_F \sim F/J\Delta(F)$ characterizing the optimal fluctuation being considered. For the left tail $R \sim \Delta$, whereas for the right tail $R \ll \Delta$, so in the latter case the effective value of D has to be estimated as $D(R) \sim \mu^2(JL/F)^{1/2} \ll D(\Delta)$, from where

$$S(F) \sim \frac{F^{5/2}}{D(R)J^{5/2}L^{1/2}} \sim \left(\frac{F}{F_*}\right)^3. \quad (14)$$

The conditions for the applicability of Eq. (14) describing the universal part of the right tail are $S(F) \gg 1$ and $R \gg x_0$, which means $F_*(L) \ll F \ll F_c(L)$. Note that here both $F_*(L)$ and $F_c(L)$ are the same as in the left tail. Another similarity to the left tail is that the fulfillment of the condition $F \gg F_*(L)$ ensures that $L \gg \Delta^2/\nu(\Delta) \sim (\Delta^3/\mu)^{1/2}$, and therefore time L is sufficient for attaining the equilibrium at the length scales of the order of Δ .¹⁷

At $F \gg F_c(L)$ one gets $R \ll x_0$, which means that the renormalization of ν and D becomes completely ineffective. Therefore, the far-right tail is described by Eq. (13) with unrenormalized parameters. Note that Eq. (13) cannot be rewritten as $S(F) \sim (F/F_*)^{5/2}$, which is quite natural, because nonuniversal part of a tail is not obliged to contain the same characteristic scale $F_*(L)$ as the universal distribution function. The appearance of the same scale in the far-left tail is not more than a coincidence.

A finite correlation length. If correlations of random potential are characterized by a finite correlation length $\xi \equiv U_0/U(0) \ll x_0$ [where now $U_0 \equiv \int dx U(x)$], expression (6) remains applicable only for $\Delta \gg \xi$ —that is, for $-F \ll F_\xi \sim (x_0/\xi)^2 F_c$. In the other regime (at $-F \gg F_\xi$) the characteristic size of $\Psi(x)$ becomes much smaller than ξ , which leads to $V(x) \propto -U(x)$, $E_0 = V(0)$, and $S \approx F^2/2U(0)L$.²¹ This means that the finiteness of ξ makes the most distant part of the nonequilibrium left tail Gaussian²² and independent of T .

With a decrease of x_0 (i.e., of temperature) the region with Gaussian behavior becomes larger and larger, and for $x_0 \ll \xi$ [i.e., $T \ll T_0 \sim (JU_0\xi)^{1/3}$] the whole nonuniversal tail must be Gaussian. In this regime the parameters of the universal part of the left tail also are modified. For $x_0 \ll \xi$ the spectral density of the equilibrium fluctuations of $u(x)$ at scales exceeding ξ can be estimated as $u_\xi^2 \xi$, where $u_\xi \sim (D\tau_\xi/\xi^3)^{1/2}$ is the characteristic velocity which is created by random force with characteristic length scale ξ during the time $\tau_\xi \sim \xi/u_\xi$ required for the breaking of such a fluctuation. This gives one an estimate $\mu \sim (D^2/\xi)^{1/3}$, substitution of which into Eq. (12) instead of the ratio D/ν leads to $S(F) \sim [-F/\tilde{F}_*]^{3/2}$, where the free-energy scale $\tilde{F}_*(L) \sim (x_0/\xi)^{2/9} F_*(L) \sim (JU_0^4/\xi^2)^{1/9} L^{1/3}$ (replacing F_*) does

not depend on T . Directly in terms of the polymer problem an analogous estimate for $\tilde{F}_*(L)$ can be obtained from scaling arguments complemented by the assumption that at low enough temperatures $\tilde{F}_*(L)$ has to be independent of T (Ref. 23) and follows also from the replica-symmetry-breaking analysis of Ref. 24.

It can be shown that $\tilde{F}_*(L)$ replaces $F_*(L)$ also in Eq. (14) describing the universal part of the right tail. On the other hand, the finiteness of ξ does not change the form of the nonuniversal part of the right tail described by Eq. (13). All this allows one to conclude that at low temperatures $P_L(F)$ becomes temperature independent, whereas the crossover between the regions of universal and nonuniversal behavior takes place at $|F| \sim \tilde{F}_c \sim (U_0^2/J\xi^4)^{1/3}L$.

Conclusion. In the present Rapid Communication we have studied the form of the free-energy distribution function $P_L(F)$ in the 1D directed polymer problem both for δ -correlated random potential and for a finite correlation length ξ . In terms of the KPZ problem the same distribution function describes the distribution of heights in the process of nonstationary growth starting from flat interface.

Our analysis has shown that for $|F| \ll F_c(L) \propto L$ both tails of $P_L(F)$ are characterized by the same free-energy scale $F_*(L)$, which confirms the universality hypothesis (1). For

$T \gg T_0 \sim (JU_0\xi)^{1/3}$ this scale is temperature dependent [see Eq. (6)], whereas at $T \ll T_0$ it saturates at a finite value. On the other hand, the exponents $\eta_- = 3/2$ and $\eta_+ = 3$ in the dependence $\ln P_L(\pm F) \propto -(F/F_*)^{\eta_{\pm}}$ are insensitive to the relation between T and T_0 . The same values of η_- and η_+ have been found by Prähofer and Spohn²⁵ for the polynuclear growth (PNG) model, which in terms of a directed polymer problem, Eq. (2), corresponds to the Poisson distribution of identical pointlike impurities and a rather peculiar limit of $J=0$ and $T=0$.^{25,26} This agreement confirms that the two models indeed belong to the same universality class.

We also have investigated the nonuniversal tails of $P_L(F)$ (at $|F| \gg F_c$). In particular, we have demonstrated that in the far-right tail $\eta_+ = 5/2$, which is in agreement with the numerical results of Kim *et al.*,²⁷ who found $\eta_+ = 2.4 \pm 0.2$. On the other hand, the form of the far-left tail depends on ξ . For $\xi=0$ it is the same ($\eta_- = 3/2$) as in the universal region, whereas for $\xi > 0$ the left tail becomes Gaussian for sufficiently large $|F|$. The nonuniversal tails in the PNG model (which, naturally, are quite different from those studied here) have been analyzed in Ref. 28.

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