Explicit solution of the optimal fluctuation problem for an elastic string in a random medium

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The free-energy distribution function of an elastic string in a quenched random potential, $P_L(F)$, is investigated with the help of the optimal fluctuation approach. The form of the far-right tail of $P_L(F)$ is found by constructing the exact solution of the nonlinear saddle-point equations describing the asymptotic form of the optimal fluctuation. The solution of the problem is obtained for two different types of boundary conditions and for an arbitrary dimension of the imbedding space $1+d$ with d from the interval $0<d<2$. The results are also applicable for the description of the far-left tail of the height distribution function in the stochastic growth problem described by the d-dimensional Kardar-Parisi-Zhang equation.

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I. INTRODUCTION

In many physical situations, the behavior of some extended manifold is determined by the competition between its internal elasticity and interaction with external random potential, which at all reasonable time scales can be treated as quenched. A number of examples of such a kind includes domain walls in magnetic materials, vortices and vortex lattices in type-II superconductors, dislocations and phase boundaries in crystals, as well as some types of biological objects. It is expected that the presence of a quenched disorder makes at least some aspects of the behavior of such systems analogous to those of other systems with quenched disorder, in particular, spin glasses.

Following Ref. [1], in the case when internal dimension of an elastic manifold is equal to 1 (that is, an object interacting with a random potential is an elastic string), the systems of such a kind are traditionally discussed under the generic name of a directed polymer in a random medium. In such a case, the problem turns out to be formally equivalent [2] to the problem of a stochastic growth described by the Kardar-Parisi-Zhang (KPZ) equation [3], which in its turn can be reduced to the Burgers equation [4] with random force (see Refs. [5,6] for reviews).

The investigation of $P_L(F)$, the free-energy distribution function for a directed polymer (of a large length L) in a random potential, was initiated by Kardar [7], who proposed an asymptotically exact method for the calculation of the moments $Z_n \equiv \overline{Z^n}$ of the distribution of the partition function Z in a $(1+1)$-dimensional system (a string confined to a plane) with a δ-correlated random potential and made an attempt of expressing the moments of $P_L(F)$ in terms of Z_n. Although soon after that, Medina and Kardar [8] (see also Refs. [5,9]) realized that the implementation of the latter task is impossible, the knowledge of Z_n allowed Zhang [10] to find the form of the tail of $P_L(F)$ at large negative F. The two attempts of generalizing the approach of Ref. [10] to other dimensions were undertaken by Zhang [11] and Kolomeisky [12].

Quite recently, it was understood [13,14] that the method of Ref. [10] allows one to study only the most distant part of the tail (the far-left tail), where $P_L(F)$ is not obliged to have the universal form $P_L(F)=P_*(F/F_*)/F_*$ (with $F_* \propto L^\omega$) it is supposed to achieve in the thermodynamic limit, $L\to\infty$. For $(1+1)$-dimensional systems, the full form of the universal distribution function is known from the ingenious exact solution of the polynuclear growth (PNG) model by Prähofer and Spohn [15]. However, there is hardly any hope of generalizing this approach to other dimensions or to other forms of random potential distribution.

One more essential step in the investigation of different regimes in the behavior of $P_L(F)$ in systems of different dimensions has been made recently [13,14] on the basis of the optimal fluctuation approach. The original version of this method was introduced in the 1960s for the investigation of the deepest part of the tail of the density of states of quantum particles localized in a quenched random potential [16–18]. Its generalization to Burgers problem has been constructed in Refs. [19,20], but for the quantities which in terms of the directed polymer problem are of no direct interest, in contrast to the distribution function $P_L(F)$ studied in Refs. [13,14]. Another accomplishment of Refs. [13,14] consists in extending the optimal fluctuation approach to the region of the universal behavior of $P_L(F)$, where the form of this distribution function is determined by an effective action with scale-dependent renormalized parameters and does not depend on how the system is described at microscopic scales.

In the current work, the results of Refs. [13,14] describing the behavior of $P_L(F)$ at the largest positive fluctuations of the free energy F (where they are not described by the universal distribution function) are rederived at a much more quantitative level by explicitly finding the form of the optimal fluctuation which is achieved in the limit of large F. This allows us not only to verify the conjectures used earlier for finding the scaling behavior of $S(F)\equiv -\ln[P_L(F)]$ in the corresponding regime, but also to establish the exact value of the numerical coefficient entering the expression for $S(F)$. For brevity, we call the part of the right tail of $P_L(F)$ studied below the far-right tail. The outlook of the paper is as follows.

In Sec. II, we formulate the continuous model which is traditionally used for the quantitative description of a directed polymer in a random medium and remind how it is related to the KPZ and Burgers problems. Section III briefly describes the saddle-point problem which has to be solved for finding the form of the most optimal fluctuation of a random potential leading to a given value of F. In Sec. IV, we construct the exact solution of the saddle-point equations introduced in Sec. III for the case when the displacement of a considered elastic string is restricted to a plane (or, in other terms, the transverse dimension of the system d is equal to 1). We do this for sufficiently large positive fluctuations of F, when the form of the solution becomes basically independent on temperature T and therefore can be found by setting T to zero.

However, the solution constructed in Sec. IV turns out to be not compatible with the required boundary conditions. Section V is devoted to describing how this solution has to be modified to become compatible with free initial condition and in Sec. VI, the same problem is solved for fixed initial condition. In both cases, we find the asymptotically exact (including a numerical coefficient) expression for $S(F)$ for the limit of large F. In Sec. VII, the results of the two previous sections are generalized for the case of an arbitrary d from the interval $0<d<2$, whereas the concluding Sec. VIII is devoted to summarizing the results.

II. MODEL

In the main part of this work, our attention is focused on an elastic string whose motion is confined to a plane. The coordinate along the average direction of the string is denoted t for the reasons which will become evident few lines below and x is string's displacement in the perpendicular direction. Such a string can be described by the Hamiltonian

$$H\{x(t)\} = \int_0^L dt \left\{ \frac{J}{2}\left[\frac{dx(t)}{dt}\right]^2 + V[t,x(t)] \right\}, \tag{1}$$

where the first term describes the elastic energy and the second one the interaction with a random potential $V(t,x)$, with L being the total length of a string along axis t. Note that the form of the first term in Eq. (1) relies on the smallness of the angle between the string and its preferred direction.

The partition function of a string which starts at $t=0$ and ends at the point (t,x) is then given by the functional integral which has exactly the same form as the Euclidean functional integral describing the motion of a quantum-mechanical particle (whose mass is given by J) in a time-dependent random potential $V(t,x)$ (with t playing the role of imaginary time and temperature T of Plank's constant $\hbar$). As a consequence, the evolution of this partition function with the increase in t is governed [2] by the imaginary-time Schrödinger equation

$$-T\dot{z} = \left[-\frac{T^2}{2J}\nabla^2 + V(t,x) \right] z(t,x). \tag{2}$$

Here and below, a dot denotes differentiation with respect to t and ∇ differentiation with respect to x.

Naturally, $z(t,x)$ depends also on the initial condition at $t=0$. In particular, fixed initial condition, $x(t=0)=x_0$, corresponds to $z(0,x)=\delta(x-x_0)$, whereas free initial condition (which implies the absence of any restrictions [9,14] on x at $t=0$) to

$$z(0,x) = 1. \tag{3}$$

Below, the solution of the problem is found for both these types of initial condition.

It follows from Eq. (2) that the evolution of the free energy corresponding to $z(t,x)$,

$$f(t,x) = -T \ln[z(t,x)], \tag{4}$$

with the increase in t is governed [2] by the KPZ equation [3]

$$\dot{f} + \frac{1}{2J}(\nabla f)^2 - \nu\nabla^2 f = V(t,x), \tag{5}$$

with the inverted sign of f, where t plays the role of time and $\nu \equiv T/2J$ of viscosity. On the other hand, the derivation of Eq. (5) with respect to x allows one to establish the equivalence [2] between the directed polymer problem and Burgers equation [4] with random potential force

$$\dot{u} + u\nabla u - \nu\nabla^2 u = J^{-1}\nabla V(t,x), \tag{6}$$

where $u(t,x)=\nabla f(t,x)/J$ plays the role of velocity. Note that in terms of the KPZ problem, the free initial condition (3) corresponds to starting the growth from a flat interface, $f(0,x)=\text{const}$, and in terms of the Burgers problem, to starting the evolution from a liquid at rest, $u(0,x)=0$.

To simplify an analytical treatment, the statistic of a random potential $V(t,x)$ is usually assumed to be Gaussian with

$$\overline{V(t,x)} = 0, \quad \overline{V(t,x)V(t',x')} = U(t-t',x-x'), \tag{7}$$

where an overbar denotes the average with respect to disorder. Although the analysis below is focused exclusively on the case of purely δ-functional correlations,

$$U(t-t',x-x') = U_0\delta(t-t')\delta(x-x'), \tag{8}$$

the results we obtain are applicable also in situations when the correlations of $V(t,x)$ are characterized by a finite correlation radius ξ because in the considered regime, the characteristic size of the optimal fluctuation grows with the increase in L and therefore for large-enough L, the finiteness of ξ is of no importance and an expression for $U(t-t',x-x')$ can be safely replaced by the right-hand side of Eq. (8) with

$$U_0 = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx U(t,x). \tag{9}$$

III. OPTIMAL FLUCTUATION APPROACH

We want to find probability of a large positive fluctuation of free energy of a string which at $t=L$ is fixed at some point $x=x_L$. It is clear that in the case of free initial condition, the result cannot depend on x_L, so for the simplification of notation, we assume below $x_L=0$ and analyze fluctuations of $F=f(L,0)-f(0,0)$. As in other situations [16–18], the probability of a sufficiently large fluctuation of F is determined by the most probable fluctuation of a random potential $V(t,x)$ leading to the given value of F.

In its turn, the most probable fluctuation of $V(t,x)$ can be found [21] by looking for the extremum of the Martin-Siggia-Rose action [22–24] corresponding to the KPZ problem,

$$S\{f,V\}=\frac{1}{U_0}\int_0^L dt\int_{-\infty}^{+\infty}dx\left\{-\frac{1}{2}V^2+V\left[\dot{f}+\frac{1}{2J}(\nabla f)^2-\nu\nabla^2 f\right]\right\}, \tag{10}$$

both with respect to $f\equiv f(t,x)$ and to a random potential realization $V\equiv V(t,x)$. The form of Eq. (10) ensures that its variation with respect to $V(t,x)$ reproduces the KPZ equation (5), whose substitution back into Eq. (10) reduces it to the expression

$$S\{V\}=\frac{1}{2U_0}\int_0^L dt\int_{-\infty}^{+\infty}dx\,V^2(t,x), \tag{11}$$

determining the probability of a given realization of a random potential, $\mathcal{P}\{V\}\propto\exp(-S\{V\})$. On the other hand, variation of Eq. (10) with respect to $f(t,x)$ shows that the time evolution of the optimal fluctuation of a random potential is governed by equation [25]

$$\dot{V}+\frac{1}{J}\nabla(V\nabla f)+\nu\nabla^2 V=0, \tag{12}$$

whose form implies that the integral of $V(t,x)$ over dx is a conserved quantity.

Our aim consist in finding the solution of Eqs. (5) and (12) satisfying condition

$$f(L,0)-f(0,0)=F, \tag{13}$$

as well as an appropriate initial condition at $t=0$. The application of this procedure corresponds to calculating the full functional integral determining $P_L(F)$ with the help of the saddle-point approximation. In the framework of this approximation, the condition (13) [which formally can be imposed by including into the functional integral determining $P_L(F)$ the corresponding δ-functional factor] leads to the appearance of the condition on $V(t,x)$ at $t=L$ [21],

$$V(L,x)=\lambda\,\delta(x), \tag{14}$$

where, however, the value of λ should be chosen to ensure the fulfillment of condition (13).

The conditions for the applicability of the saddle-point approximation for the analysis of the far-right tail of $P_L(F)$ are given by $S\gg 1$ and $F\gg JU_0^2L/T^4$. The origin of the former inequality is evident, whereas the fulfillment of the latter one ensures the possibility to neglect the renormalization of the parameters of the system by small-scale fluctuations [26]. We also assume that $F\gg T$, which ensures that the characteristic length scale of the optimal fluctuation is sufficiently large to neglect the presence of viscous terms in Eqs. (5) and (12) [26]. This allows us to replace Eqs. (5) and (12) by

$$\dot{f}+\frac{1}{2J}(\nabla f)^2=V, \tag{15a}$$

$$\dot{V}+\frac{1}{J}\nabla(V\nabla f)=0, \tag{15b}$$

which formally corresponds to considering the original (directed polymer) problem at zero temperature, $T=0$, where the free energy of a string is reduced to its ground-state energy. In accordance with that, in the $T=0$ limit, $f(L,0)$ is given by the minimum of Hamiltonian (1) on all string's configurations $x(t)$ which at $t=0$ satisfy a chosen initial condition and at $t=L$ end up at $x(L)=0$.

Exactly like Eq. (12), Eq. (15b) implies that $V(t,x)$ behaves itself like a density of a conserved quantity, but takes into account only the nondissipative component to the flow of V given by Vu, where

$$u\equiv u(t,x)\equiv J^{-1}\nabla f(t,x) \tag{16}$$

plays the role of velocity. Naturally, for $\nu=0$, the time evolution of u is governed by the nondissipative version of the force-driven Burgers equation (6),

$$\dot{u}+u\nabla u=\nabla V/J. \tag{17}$$

IV. EXACT SOLUTION OF THE SADDLE-POINT EQUATIONS

It is clear from symmetry that in the optimal fluctuation we are looking for, both $f(t,x)$ and $V(t,x)$ have to be even functions of x. After expanding them at $x=0$ in Taylor series, it is easy to verify that an exact solution of Eqs. (15) can be constructed by keeping in each of these expansions only the first two terms

$$f(t,x)=J[A(t)-B(t)x^2], \tag{18a}$$

$$V(t,x)=J[C(t)-D(t)x^2]. \tag{18b}$$

Substitution of Eqs. (18) into Eqs. (15) gives then a closed system of four equations

$$\dot{A}=C, \tag{19a}$$

$$\dot{B}=2B^2+D, \tag{19b}$$

$$\dot{C}=2BC, \tag{20a}$$

$$\dot{D}=6BD, \tag{20b}$$

which determines the evolution of coefficients A, B, C, and D with the increase in t.

It is easy to see that with the help of Eq. (20b), $D(t)$ can be expressed in terms of $B(t)$, which allows one to transform Eq. (19b) into a closed equation for $B(t)$,

$$\dot{B}=2B^2+D(t_0)\exp\left[6\int_{t_0}^{t}dt'\,B(t')\right]. \tag{21}$$

After making a replacement

$$B(t)=-\frac{\dot{\varphi}}{2\varphi}, \tag{22}$$

Eq. (21) is reduced to an equation of the Newton's type,

$$\ddot{\phi}+\frac{\alpha}{\phi^2}=0, \tag{23}$$

where $\alpha \equiv 2D(t)\phi^3(t)$ is an integral of motion which does not depend on t.

Equation (23) can be easily integrated which allows one to ascertain that its general solution can be written as $\phi(t)=\phi_0\Phi[(t-t_0)/L_*]$, where t_0 and $\phi_0 \equiv \phi(t_0)$ are arbitrary constants,

$$L_* = \frac{\pi}{4[D(t_0)]^{1/2}} \tag{24}$$

plays the role of the characteristic time scale, and $\Phi(\eta)$ is an even function of its argument implicitly defined in the interval $-1 \le \eta \le 1$ by equation

$$\sqrt{\Phi(1-\Phi)} + \arccos\sqrt{\Phi} = \frac{\pi}{2}|\eta|. \tag{25}$$

With the increase of $|\eta|$ from 0 to 1, $\Phi(\eta)$ monotonically decreases from 1 to 0. In particular, on approaching $\eta = \pm 1$, the behavior of $\Phi(\eta)$ is given by

$$\Phi(\eta) \approx [(3\pi/4)(1-|\eta|)]^{2/3}. \tag{26}$$

Since it is clear from the form of Eq. (22) that the constant ϕ_0 drops out from the expression for $B(t)$, one without the loss of generality can set $\phi_0=1$ and

$$\phi(t) = \Phi\left(\frac{t-t_0}{L_*}\right). \tag{27}$$

The functions $A(t)$, $B(t)$, $C(t)$, and $D(t)$ can be then expressed in terms of $\phi \equiv \phi(t)$ as

$$A(t) = A_0 + \mathrm{sgn}(t-t_0)\frac{C_0}{D_0^{1/2}}\arccos\sqrt{\phi}, \tag{28a}$$

$$B(t) = \mathrm{sgn}(t-t_0)[D_0(1-\phi)/\phi^3]^{1/2}, \tag{28b}$$

$$C(t) = C_0/\phi, \tag{28c}$$

$$D(t) = D_0/\phi^3, \tag{28d}$$

where $A_0=A(t_0)$, $C_0=C(t_0)$, and $D_0=D(t_0)$.

Thus we have found an exact solution of Eqs. (15) in which $f(t,x)$ is maximal at $x=0$ (for $t>t_0$) and the value of $f(t,0)$ monotonically grows with the increase in t. However, the optimal fluctuation also have to satisfy particular boundary conditions. The modifications of the solution given by Eqs. (27) and (28) compatible with two different types of initial conditions, free and fixed, are constructed in Secs. V and VI, respectively.

V. FREE INITIAL CONDITION

When the initial end point of a polymer (at $t=0$) is not fixed (that is, is free to fluctuate), the boundary condition at $t=0$ can be written as $z(0,x)=1$ or

$$f(0,x)=0.$$

Apparently, this condition is compatible with Eq. (18a) and in terms of functions $A(t)$ and $B(t)$ corresponds to

$$A(0)=0, \quad B(0)=0, \tag{29}$$

from where $\dot{\phi}(0)=0$ and $t_0=0$. However, it is clear that the solution described by Eqs. (27)–(29) cannot be the optimal one because it does not respect condition (14) which has to be fulfilled at $t=L$. Moreover, this solution corresponds to an infinite action and the divergence of the action is coming from the regions where potential $V(t,x)$ is negative, which evidently cannot be helpful for the creation of a large positive fluctuation of $f(t,0)$.

From the form of Eqs. (1) and (11), it is clear that any region where $V(t,x)<0$ cannot increase the energy of a string but makes a positive contribution to the action. Therefore, in a really optimal fluctuation with $F>0$, potential $V(t,x)$ should be either positive or zero. In particular, since just the elastic energy of any configuration $x(t)$ which somewhere crosses or touches the line

$$x_*(t) = [2F(L-t)/J]^{1/2} \tag{30}$$

and at $t=L$ ends up at $x(L)=0$ is already larger than F, there is absolutely no reason for $V(t,x)$ to be nonzero at least for $|x|>x_*(t)$.

It turns out that the exact solution of the saddle-point equations (15) in which potential $V(t,x)$ satisfies boundary condition (14) and constraint $V(t,x)\ge 0$ can be constructed on the basis of the solution found in Sec. IV just by cutting the dependences (18) at the points

$$x = \pm X(t), \quad X(t) \equiv \left[\frac{C(t)}{D(t)}\right]^{1/2} = \left(\frac{C_0}{D_0}\right)^{1/2}\phi(t), \tag{31}$$

where $V(t,x)$ is equal to zero, and replacing them at $|x|>X(t)$ by a more trivial solution of the same equations with $V(t,x)\equiv 0$ which at $x=\pm X(t)$ has the same values of $f(t,x)$ and $u(t,x)$ as the solution at $|x|\le X$. Such a replacement can be done because the flow of V through the moving point $x=X(t)$ in both solutions is equal to zero. In accordance with that, the integral of $V(t,x)$ over the interval $-X(t)<x<X(t)$ does not depend on t. It is clear from Eq. (31) that $X(t)$ is maximal at $t=t_0$ and at $t>t_0$ monotonically decreases with the increase of t.

The form of $f(t,x)$ at $|x|>X(t)$ is then given by

$$f(t,x) = f[t,X(t)] + J\int_{X(t)}^{|x|} dx'\, u_0(t,x'), \tag{32}$$

where $u_0(t,x)$ is the solution of Eq. (17) with zero right-hand side in the region $x>X(t)$ which at $x=X(t)$ satisfies boundary condition

$$u_0[t,X(t)] = v(t). \tag{33}$$

In Eq. (33), we have taken into account that in the solution constructed in Sec. IV $u[t,X(t)]=-2B(t)X(t)$ coincides with

$$v(t) = \frac{dX}{dt} = \left(\frac{C_0}{D_0}\right)^{1/2} \dot{\phi} = -2\sqrt{C_0(\phi^{-1}-1)}, \qquad (34)$$

the velocity of the point $x=X(t)$. This immediately follows from Eq. (22) and ensures that the points where spacial derivatives of $u(t,x)$ and $V(t,x)$ have jumps always coincide with each other. It is clear from Eq. (34) that $v(t_0)=0$, whereas at $t>t_0$, the absolute value of $v(t)<0$ monotonically grows with the increase in t.

Since Eq. (17) with vanishing right-hand side implies that the velocity of any Lagrangian particle does not depend on time, its solution satisfying boundary condition (33) can be written as

$$u_0(t,x) = v[\tau(t,x)], \qquad (35)$$

where function $\tau(t,x)$ is implicitly defined by equation

$$x = X(\tau) + (t-\tau)v(\tau). \qquad (36)$$

Monotonic decrease of $v(\tau)<0$ with the increase in τ ensures that in the interval $X(t)<x<X_0\equiv X(t_0)$, Eq. (36) has a well-defined and unique solution which at fixed t monotonically decreases from t at $x=X(t)$ to 0 at $x=X_0$. In accordance with that, $u_0(t,x)$ as a function of x monotonically increases from $v(t)<0$ at $x=X(t)$ to 0 at $x=X_0$. For free initial condition (implying $t_0=0$), the form of the solution at $x>X_0$ remains the same as in the absence of optimal fluctuation, that is, $u_0[t,x>X_0]\equiv 0$. The fulfillment of the inequality $\partial u_0(t,x)/\partial x \leq 0$ in the interval $x>X(t)$ demonstrates the absence of any reasons for the formation of additional singularities (such as shocks), which confirms the validity of our assumption that the form of the solution can be understood without taking into account viscous terms in saddle-point equations (5) and (12).

Substitution of Eqs. (35) and (36) into Eq. (32) and application of Eqs. (15a) and (17) allow one to reduce Eq. (32) to

$$f(t,x) = \frac{J}{2}\int_0^{\tau(t,x)} d\tau'(t-\tau')\frac{dv^2(\tau')}{d\tau'} \qquad (37)$$

from where it is immediately clear that on approaching $x=X_0$, where $\tau(t,x)$ tends to zero, $f(t,x)$ also tends to zero, so that at $|x|>X_0$, the free energy $f(t,x)$ is equal to zero (that is, remains exactly the same as in the absence of optimal fluctuation). However, for our purposes, the exact form of the solution at $|x|>X(t)$ is of no particular importance because this region does not contribute anything to the action.

It is clear that the compatibility of the constructed solution with condition (14) is achieved when the interval $[-X(t),X(t)]$ where the potential is nonvanishing shrinks to a point. This happens when the argument of function Φ in Eq. (27) is equal to 1, that is when

$$t_0 + L_* = L, \qquad (38)$$

which for $t_0=0$ corresponds to $L_*=L$ and

$$D_0 = \left(\frac{\pi}{4L}\right)^2. \qquad (39)$$

On the other hand, Eq. (28a) with $A_0=0$ gives $A(L) =(\pi/2)C_0/D_0^{1/2}=2LC_0$. With the help of the condition $A(L)=F/J$ following from Eq. (13), this allows one to conclude that

$$C_0 = \frac{F}{2JL}. \qquad (40)$$

Thus, for free initial condition, the half width of the region where the optimal fluctuation of a random potential is localized is equal to

$$X_0 = \left(\frac{C_0}{D_0}\right)^{1/2} = \frac{2}{\pi}\left(\frac{2FL}{J}\right)^{1/2} \qquad (41)$$

at $t=0$ (when it is maximal) and monotonically decreases to zero as $X(t)=X_0\Phi(t/L)$ when t increases to L. On the other hand, $V(t,0)$, the amplitude of the potential, is minimal at $t=0$ (when it is equal to $F/2L$) and monotonically increases to infinity. In the beginning of this section, we have argued that $V(t,x)$ has to vanish at least for $|x|>x_*(t)=[2F(L-t)/J]^{1/2}$ and indeed it can be checked that $X(t)<x_*(t)$ at all t, the maximum of the ratio $X(t)/x_*(t)$ being approximately equal to 0.765.

In the case of free initial condition, the optimal fluctuation of a random potential at $T=0$ has to ensure that $E(x_0)$, the minimum of $H\{x(t)\}$ for all string's configurations with $x(0)=x_0$ and $x(L)=0$, for all values of x_0 should be equal or larger than F. In particular, for any x_0 from the interval $|x_0|<X_0$ where the potential is nonzero, the corresponding energy $E(x_0)$ has to be exactly equal to F, otherwise there would exist a possibility to locally decrease the potential without violating the condition $E(x_0)\geq F$.

The configuration of a string, $x(t)$, which minimizes $H\{x(t)\}$ in the given realization of a random potential for the given values of x_0 and $x(L)$, at $0<x<L$ has to satisfy equation

$$-J\frac{d^2x}{dt^2} + \frac{\partial V(t,x)}{\partial x} = 0, \qquad (42)$$

which is obtained by the variation of Hamiltonian (1) with respect to x. It is not hard to check that for the optimal fluctuation found above, the solution of this equation for an arbitrary x_0 from the interval $-X_0<x_0<X_0$ can be written as

$$x(t) = \frac{x_0}{X_0}X(t). \qquad (43)$$

All these solutions have the same energy, $E(x_0)=F$.

The value of the action corresponding to the optimal fluctuation can be then found by substituting Eqs. (18b), (28c), and (28d) into the functional (11), where the integration over dx should be restricted to the interval $-X(t)<x<X(t)$, which gives

$$S_{\text{free}} = \frac{8}{15}\frac{C_0^{5/2}}{D_0^{1/2}}\frac{J^2}{U_0}\int_0^L \frac{dt}{\phi(t)} = \frac{4\pi}{15}\frac{C_0^{5/2}}{D_0}\frac{J^2}{U_0}. \qquad (44)$$

The integral over dt in Eq. (44) can be calculated with the help of replacement $dt/\phi=d\phi/(\phi\dot{\phi})$ and is equal to $\pi/2D_0^{1/2}$. Substitution of relations (39) and (40) allows one to rewrite Eq. (44) in terms of the parameters of the original system as

$$S_{\text{free}}(F,L) = K\frac{F^{5/2}}{U_0 J^{1/2} L^{1/2}}, \quad K = \frac{8\sqrt{2}}{15\pi}. \tag{45}$$

The exponents entering Eq. (45) have been earlier found in Ref. [13] from the scaling arguments based on the assumption that for large L, the form of the optimal fluctuation involves a single relevant characteristic length scale with the dimension of x which algebraically depends on the parameters of the system (including L) and grows with the increase of L [27]. The analysis of this section has explicitly confirmed this assumption and has allowed us to find the exact value of the numerical coefficient K.

Since the characteristic length scale of the solution we constructed is given by $X_0 \equiv X(t_0) \sim (FL/J)^{1/2}$, the neglect of viscosity ν remains justified as long as the characteristic relaxation time corresponding to this length scale $\tau_{\text{rel}} \sim X_0^2/\nu \sim FL/T$ is much larger than the time scale of this solution L, which corresponds to

$$F \gg T. \tag{46}$$

Another condition for the validity of Eq. (45) is the condition for the direct applicability of the optimal fluctuation approach. One can disregard any renormalization effects as long as the characteristic velocity inside optimal fluctuation is much larger [14] than the characteristic velocity of equilibrium thermal fluctuations at the length scale $x_c \sim T^3/JU_0$, the only characteristic length scale with the dimension of x which exists in the problem with δ-functional correlations (that is, can be constructed from T, J, and U_0). In terms of F, this condition reads

$$F \gg U_0^2 JL/T^4. \tag{47}$$

It is easy to check that the fulfillment of conditions (46) and (47) automatically ensures $S \gg 1$, which also is a necessary condition for the applicability of the saddle-point approximation.

For $L \gg L_c$, where $L_c \sim T^5/JU_0^2$ is the only characteristic length scale with the dimension of L which exists in the problem with δ-functional correlations, condition (46) automatically follows from condition (47) which can be rewritten as $F \gg (L/L_c)T$. Thus, for a sufficiently long string (with $L \gg L_c$), the only relevant restriction on F is given by Eq. (47).

VI. FIXED INITIAL CONDITION

When both end points of a string are fixed [$x(0)=x_0$, $x(L)=x_L$], one without the loss of generality can consider the problem with $x_0 = x_L$. Due to the existence of so-called tilting symmetry [28], the only difference between the problems with $x_0 = x_L$ and $x_0 \neq x_L$ consists in the shift of the argument of $P_L(F)$ by $\Delta F \equiv J(x_L - x_0)^2/2L$. For this reason, we consider below only the case $x_0 = x_L = 0$.

When a string is fastened at $t=0$ to the point $x=0$, in terms of $z(t,x)$, the boundary condition at $t=0$ can be written as $z(0,x) \propto \delta(x)$. In such a case, the behavior of $f(t,x)$ at $t \to 0$ is dominated by the elastic contribution to energy, which allows one to formulate the boundary condition in terms of $f(t,x)$ as [14]

$$\lim_{t\to 0}[f(t,x) - f^{(0)}(t,x)] = 0, \tag{48}$$

where $f^{(0)}(t,x) = Jx^2/2t$ is the free energy of the same system in the absence of a disorder. Since we are explicitly analyzing only the $T \to 0$ limit, we omit the linear in T contribution to the expression for $f^{(0)}(t,x)$ which vanishes in this limit. The fulfillment of condition (48) can be ensured, in particular, by setting

$$f(\varepsilon,x) = f^{(0)}(\varepsilon,x) = Jx^2/2\varepsilon, \tag{49}$$

which corresponds to suppressing the noise in the interval $0 < t < \varepsilon$, and afterwards taking the limit $\varepsilon \to 0$. Naturally, the free initial condition also can be written in the form (48) but with $f^{(0)}(t,x) = 0$.

Quite remarkably, initial condition (49) is compatible with the structure of the solution constructed in Sec. IV and in terms of functions $A(t)$ and $B(t)$ corresponds to

$$A(\varepsilon) = 0, \quad B(\varepsilon) = -1/2\varepsilon. \tag{50}$$

Substitution of Eqs. (24), (26), and (27) into Eq. (28b) allows one to establish that for $\varepsilon \ll L_*$, the condition $B(\varepsilon) = -1/2\varepsilon$ corresponds to

$$t_0 - L_* \approx \varepsilon/3. \tag{51}$$

Exactly like in the case of free initial condition (see Sec. V), we have to assume that at $|x| > X(t)$, the dependences (18a) and (18b) are replaced, respectively, by Eq. (32) and $V(t,x) = 0$. The compatibility with condition (14) is achieved then when the interval $-X(t) < x < X(t)$ where the potential is nonvanishing shrinks to a point, the condition for which is given by Eq. (38). A comparison of Eq. (38) with Eq. (51) allows one to conclude that for initial condition (50) $L_* \approx L/2 - \varepsilon/6$ and $t_0 \approx L/2 + \varepsilon/6$, which after taking the limit $\varepsilon \to 0$ gives

$$L_* = L/2, \quad t_0 = L/2. \tag{52}$$

This unambiguously defines the form of the solution for the case of fixed initial condition.

In this solution, the configuration of $V(t,x)$ is fully symmetric not only with respect to the change of the sign of x but also with respect to replacement

$$t \Rightarrow L - t. \tag{53}$$

The origin of this property is quite clear. In terms of an elastic string, the problem we are analyzing now is fully symmetric with respect to replacement (53), therefore it is quite natural that the spacial distribution of the potential in the optimal fluctuation also has to have this symmetry.

Since we are considering the limit of zero temperature when the free energy of a string is reduced to its energy, which in its turn is just the sum of the energies of the two halves of the string, the form of the potential $V(t,x)$ in the symmetric optimal fluctuation can be found separately for each of the two halves after imposing free boundary condition at $t = L/2$. This form can be described by Eqs. (27), (28c), and (28d) with $L_* = t_0 = L/2$, where the values of C_0 and D_0 can be obtained from Eqs. (40) and (39), respectively, by replacement

$$F \Rightarrow F/2, \quad L \Rightarrow L/2. \tag{54}$$

The value of the action corresponding to the optimal fluctuation can be then found by making the same replacement in Eq. (45) and multiplying the result by the factor of 2,

$$S_{\mathrm{fix}}(F,L) = 2S_{\mathrm{free}}(F/2,L/2) = \frac{1}{2}S_{\mathrm{free}}(F,L). \tag{55}$$

Naturally, the conditions for the applicability of Eq. (55) are the same as for Eq. (45) (see the two last paragraphs of Sec. V). The claim that the optimal fluctuation is symmetric with respect to replacement (53) and therefore both halves of the string make equal contributions to its energy can be additionally confirmed by noting that the sum

$$S_{\mathrm{free}}(F',L/2) + S_{\mathrm{free}}(F-F',L/2) \tag{56}$$

is minimal when $F'=F-F'=F/2$.

Like in the case of free initial condition, the form of the optimal fluctuation is such that the whole family of extremal string's configurations satisfying Eq. (42) is characterized by the same value of energy, $E\equiv H\{x(t)\}=F$. Formally, this family again can be described by Eq. (43) where x_0 now should be understood not as $x(0)$ but more generally as $x(t_0)$.

VII. GENERALIZATION TO OTHER DIMENSIONALITIES

The same approach can be applied in the situation when polymer's displacement is not a scalar quantity but a d-dimensional vector $\mathbf{x}$. In such a case, the expressions for the action and for the saddle-point equations retain their form, where now operator ∇ should be understood as vector gradient. A spherically symmetric solution of Eqs. (15) can be then again found in the form (18) with $x^2\equiv\mathbf{x}^2$.

For arbitrary d substitution of Eqs. (18) into Eqs. (15) reproduces Eqs. (19) in exactly the same form, whereas Eqs. (20) are replaced by

$$\dot{C} = 2dBC, \tag{57a}$$

$$\dot{D} = (4+2d)BD. \tag{57b}$$

A general solution of Eqs. (19) and (57) can be then written as

$$A(t) = A_0 + \mathrm{sgn}(t-t_0)\frac{C_0 I_-(\phi,d)}{2(dD_0)^{1/2}}, \tag{58a}$$

$$B(t) = \mathrm{sgn}(t-t_0)\left[\frac{D_0}{d}\frac{1-\phi^d}{\phi^{2+d}}\right]^{1/2}, \tag{58b}$$

$$C(t) = C_0/\phi^d, \tag{58c}$$

$$D(t) = D_0/\phi^{2+d}, \tag{58d}$$

where

$$\phi \equiv \phi(t) = \Phi\left(\frac{t-t_0}{L_*}\right), \tag{59}$$

with

$$L_* = \frac{I_+(0,d)}{2(dD_0)^{1/2}} \tag{60}$$

and $\Phi(\eta)$ is an even function of its argument implicitly defined in the interval $-1\leq\eta\leq1$ by equation

$$I_+(\Phi,d)] = I_+(0,d)|\eta|. \tag{61}$$

Here, $I_\pm(\phi,d)$ stands for the integral

$$I_\pm(\phi,d) = \int_{\phi^d}^{1} dq \frac{q^{1/d-1\pm1/2}}{(1-q)^{1/2}}, \tag{62}$$

in accordance with which $I_\pm(0,d)$ is given by the Euler beta function

$$I_\pm(0,d) = B\left(\frac{1}{2},\frac{1}{d}\pm\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{d}\pm\frac{1}{2}\right)}{\Gamma\left(\frac{1}{d}+\frac{1}{2}\pm\frac{1}{2}\right)}. \tag{63}$$

From the form of Eqs. (61) and (62), it is clear that with the increase of $|\eta|$ from 0 to 1, the function $\Phi(\eta)$ monotonically decreases from 1 to 0. It is not hard to check that at $d=1$, Eqs. (58) and (61) are reduced to Eqs. (28) and (25), respectively.

Exactly like in the case $d=1$, for free initial condition, one gets $t_0=0$, $L_*=L$, $A_0=0$, and $A(L)=F/J$ from where

$$C_0 = \frac{2-d}{2}\frac{F}{JL}, \quad D_0 = \frac{1}{d}\left[\frac{I_+(0,d)}{2L}\right]^2. \tag{64}$$

On the other hand, Eq. (44) is replaced by

$$S_{\mathrm{free}} = \frac{4\Omega_d}{d(d+2)(d+4)}\frac{C_0^{1+d/2}}{D_0^{d/2}}\frac{JF}{U_0}, \tag{65}$$

where $\Omega_d=2\pi^{d/2}/\Gamma(d/2)$ is the area of a d-dimensional sphere. Substitution of Eqs. (64) into Eq. (65) then gives

$$S_{\mathrm{free}}(F,L) = K_d\frac{F^{2+d/2}}{U_0 J^{d/2}L^{1-d/2}}, \tag{66a}$$

with

$$K_d = \frac{8(2-d)^{1+d/2}(2d)^{d/2-1}}{(d+2)(d+4)\Gamma(d/2)}\left[\frac{\Gamma(1/d+1)}{\Gamma(1/d+1/2)}\right]^d. \tag{66b}$$

Naturally, at $d=1$ numerical coefficient K_d coincides with coefficient K in Eq. (45). Like in the case of $d=1$, the value of the action for fixed initial condition, $\mathbf{x}(t=0)=0$, can be found by making in Eq. (66a) replacement Eq. (54) and multiplying the result by the factor of 2, which gives

$$S_{\mathrm{fix}}(F,L) = 2S_{\mathrm{free}}(F/2,L/2) = \left(\frac{1}{2}\right)^d S_{\mathrm{free}}(F,L). \tag{67}$$

The exponents entering Eq. (66a) and determining the dependence of S_{free} on the parameters of the system have been earlier found in Ref. [14] from the scaling arguments based on the assumption that for large L, the form of the optimal fluctuation involves a single relevant characteristic length scale with the dimension of $\mathbf{x}$ which algebraically depends on the parameters of the system (including L) and grows with the increase of L [27]. However, the analysis of this section reveals that this length scale, $X_0=(C_0/D_0)^{1/2}$, tends to zero when d approaches 2 from below, as well as the value of the action given by Eqs. (66).

This provides one more evidence that at $d\geq 2$, the problem with purely delta-functional correlations of a random potential becomes ill defined [29] and has to be regularized in some way, for example, by introducing a finite correlation length for the random potential correlator. In such a situation, the geometrical size of the optimal fluctuation is determined by this correlation length [14] and its shape is not universal, that is, depends on the particular form of the random potential correlator. Thus, the range of the applicability of Eqs. (66) is restricted to $0<d<2$ and includes only one physical dimension, $d=1$.

VIII. CONCLUSION

In the current work, we have investigated the form of $P_L(F)$, the distribution function of the free energy of an elastic string with length L subject to the action of a random potential with a Gaussian distribution. This has been done in the framework of the continuous model traditionally used for the description of such systems, Eq. (1). Our attention has been focused on the far-right tail of $P_L(F)$, that is on the probability of a very large positive fluctuation of free energy F in the regime when this probability is determined by the probability of the most optimal fluctuation of a random potential leading to the given value of F.

We have constructed the exact solution of the nonlinear saddle-point equations describing the asymptotic form of the optimal fluctuation in the limit of large F when this form becomes independent of temperature. This has allowed us to find not only the scaling from of $S(F)=-\ln[P_L(F)]$ but also the value of the numerical coefficient in the asymptotic expression for $S(F)$.

The solution of the problem has been obtained for two different types of boundary conditions (corresponding to fixing either one or both end points of a string) and for an arbitrary dimension of the imbedding space $1+d$ with d from the interval $0<d<2$ (d being the dimension of the displacement vector). Quite remarkably, in both cases, the asymptotic expressions for $S(F)$, Eqs. (66) and (67), are rather universal. In addition to being independent of temperature, they are applicable not only in the case of δ-correlated random potential explicitly studied in this work, but also (for a sufficiently large L) in the case of potential whose correlations are characterized by a finite correlation radius. Note that our results cannot be compared to those of Brunet and Derrida [30] because these authors have considered a very specific regime when the transverse size of a system (with cylindrical geometry) scales in a particular way with its length L.

Due to the existence of the equivalence [2] between the directed polymer and KPZ problems, the distribution function of the directed polymer problem in situation when only one of the end points is fixed (and the other is free to fluctuate) describes also the fluctuations of height [13,14] in the d-dimensional KPZ problem in the regime of nonstationary growth which have started from a flat configuration of the interface, L being the total time of the growth. The only difference is that the far-right tail of $P_L(F)$ studied in this work in the traditional notation of the KPZ problem [3] corresponds to the far-left tail of the height distribution function. In terms of the KPZ problem, the independence of the results on temperature is translated into their independence on viscosity.

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