Surface spin waves in ^{3}He-B

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A dispersion equation for surface spin waves in the isotropic superfluid phase of ^{3}He (the B-phase) is derived and investigated. It is shown that the spectrum has an end point at small wave vectors k. The velocities of spin waves of various polarizations in a thin ^{3}He-B layer are calculated in the limit of large k.

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The order parameter for the superfluid phases of ^{3}He is of the form $\Delta(p,T)A^{ij}$, where $\Delta(p,T)$ is a scalar factor that depends on pressure and temperature and vanishes on the superfluid transition curve, while A^{ij} is a normalization factor that expresses the angular dependence and transforms like a vector with respect to the index i in rotations of ordinary (orbital) space, and like a vector, but in terms of the index j, in spin-space rotations. The invariant properties of the matrix A^{ij} are determined by the particular superfluid phase of ^{3}He involved, and its form depends, of course, also on the arrangements of the coordinate axes in orbital and spin spaces.

In particular, for the isotropic superfluid phase of ^{3}He (the B phase), A^{ij} is a real orthogonal matrix that sets a one-to-one correspondence between the vectors in spin and orbital spaces. By rotating the coordinate axes of the spin space relative to coordinate axes of orbital space, we can always transform the matrix A^{ij} into the unit matrix δ_{ij}.

It is known (see, e.g., Ref. 1) that with this choice of the coordinate axes the linearized equations of the B-phase spin dynamics, expressed in terms of the small rotation angle u^k that describes the deviation of A^{ij} from its equilibrium value δ_{ij}:

$$A^{ij}=\delta_{ij}+\varepsilon^{ikl}\delta_{lj}u^k,$$

assume the simplest form and reduce, if the dipole interaction is neglected, to an ordinary vector wave equation

$$\partial^2\mathbf{u}/\partial t^2=c_t^2\Delta\mathbf{u}+(c_l^2-c_t^2)\,\mathrm{grad\,div}\,\mathbf{u}. \qquad (1)$$

Starting only from the fact that an equation of exactly the same form (where $\mathbf{u}$ is the strain vector) describes the elastic oscillations of an isotropic solid, Marchenko[2] reached the conclusion that surface spin waves analogous to Rayleigh waves in a solid should exist on a flat ^{3}He-B boundary.

In fact, since the boundary condition for ^{3}He-B (the vanishing of the spin current through the boundary is expressed in terms of $\mathbf{u}$ (Ref. 1):

$$c_t^2 n^k\frac{\partial u^i}{\partial x^k}+\frac{1}{2}(c_l^2-c_t^2)n^k\frac{\partial u^k}{\partial x^i}+\frac{1}{2}(c_l^2-c_t^2)\,n^i\frac{\partial u^k}{\partial x^k}=0, \qquad (2)$$

does not coincide with the boundary condition for an elastic solid (vanishing of the normal components of the elastic-stress tensor) [Ref. 3, §24]:

$$c_t^2 n^k\frac{\partial u^i}{\partial x^k}+c_t^2 n^k\frac{\partial u^k}{\partial x^i}+(c_l^2-2c_t^2)\,n^i\frac{\partial u^k}{\partial x^k}=0, \qquad (3)$$

the question of the existence of surface spin waves in ^{3}He-B cannot be reduced to a simple analogy with Rayleigh waves in a solid and calls for a separate study, to which this paper is devoted. It is also of interest to take the dipole interaction into consideration.

DERIVATION OF THE DISPERSION EQUATION FOR SURFACE SPIN WAVES

We consider first ^{3}He-B in a half-space bounded by a rigid wall with which it does not interact. Let $\mathbf{n}$ be the outward normal to the boundary. We confine ourselves to the case of zero temperature and neglect the difference between the coefficients in the gradient terms in the expansion of the potential energy from their values on the curve of the transition into the superfluid state.

If we disregard the dipole interaction, the minimum of the energy of such a system is reached in any of the states with a matrix A^{ij} that is homogeneous over the volume. When the dipole interaction in the volume of the liquid is taken into account, this degeneracy is partly lifted, and the minimum of the energy is reached only for states at which the orthogonal matrix homogeneous over the volume, A^{ij}, is the matrix that effects a rotation through an angle arccos$(-\frac{1}{4})$ around an arbitrary axis.[4] Allowance for the surface dipole interaction lifts also the degeneracy in the rotation-axis directions and separates, as the states with minimum energy, two states for which the angle of rotation through the indicated angle is parallel to $\mathbf{n}$, and the rotation directions are different.[5] The form of the equations obtained subsequently does not depend on which of these states is taken to be the unperturbed state of the system.

To simplify the equations it makes sense to assume that the coordinate axes in spin space are so rotated that $A^{ij}=\delta_{ij}$ in the chosen unperturbed state. We write down the Lagrangian of the system in the quadratic approximation, expressing in terms of $\mathbf{u}$ the kinetic energy, as well as the gradient and dipole terms in the potential energy[1]:

$$L=\int\left\{\frac{\chi}{2\gamma^2}\left[\left(\frac{\partial u^i}{\partial t}\right)^2-c_t^2\frac{\partial u^i}{\partial x^k}\frac{\partial u^i}{\partial x^k}-\frac{1}{2}(c_l^2-c_t^2)\left(\frac{\partial u^i}{\partial x^k}\frac{\partial u^k}{\partial x^i}+\frac{\partial u^i}{\partial x^i}\frac{\partial u^k}{\partial x^k}\right)\right]-\frac{1}{2}g_D(n^iu^i)^2\right\}dV-\frac{1}{2}\int b^{ij}u^iu^j\,dS. \qquad (4)$$

It was taken into account here that the anisotropy axis is parallel to $\mathbf{n}$: g_D and $b^{ij}\sim g_D\,\xi_0$ are constants that characterize the dipole interaction. It is clear from symmetry that the tensor b^{ij} is of the form

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$b_t(\delta_{ij}-n^in^j)+b_nn^in^j$.

Varying (4), we can write down the Lagrange equation

$$\frac{\partial^2\mathbf{u}}{\partial t^2}=c_t^2\,\Delta\mathbf{u}+(c_l^2-c_t^2)\,\mathrm{grad\,div\,}\mathbf{u}-c_t^2\,\xi_D^{-2}\mathbf{n}(\mathbf{nu}) \qquad (5)$$

and the boundary condition

$$c_t^2n^k\frac{\partial u^i}{\partial x^k}+\frac{1}{2}(c_l^2-c_t^2)\left(n^k\frac{\partial u^k}{\partial x^i}+n^i\frac{\partial u^k}{\partial x^k}\right)+c_t^2\,\xi_0\xi_D^{-2}B^{ik}u^k=0, \qquad (6)$$

which are generalizations of Eqs. (1) and (2) with allowance for the dipole interaction. Here

$$\xi_D=(\chi c_t^2/\gamma^2g_D)^{1/2}\gg\xi_0,$$

$$B^{ik}=\gamma^2\xi_D^2b^{ik}/\chi c_t^2\xi_0=B_t(\delta_{ik}-n^in^k)+B_nn^in^k,\quad B_t,\,B_n\sim1.$$

To find a surface-wave-type simultaneous solution of (5) and (6), we must take their Fourier transforms with respect to time (with frequency ω) and with respect to the coordinates along the boundary plane (with wave vector $\mathbf{k}\perp\mathbf{n}$), after which $\mathbf{u}_{\omega\mathbf{k}}$ is a function of only the coordinate $z=\mathbf{n}\cdot\mathbf{r}$; we shall omit hereafter the subscripts ω and $\mathbf{k}$ of $\mathbf{u}_{\omega\mathbf{k}}$.

A solution of the surface-wave type should be constructed as a linear combination of different linearly independent solutions of Eq. (5); these solutions attenuate in the interior of the liquid (as $z\to-\infty$) and are obviously of the form $\mathbf{u}_0\exp(\varkappa z)$ with $\varkappa>0$. The equation obtained for $\varkappa$, as the condition for the compatibility which follows from (5) for the system of scalar linear algebraic equations, breaks up into two equations (biquadratic and quadratic). The solutions of the biquadratic equation

$$c_t^2c_l^2\varkappa^4-[c_t^2c_l^2(\xi_D^{-2}+2k^2)-(c_t^2+c_l^2)\,\omega^2]\,\varkappa^2+[(c_l^2k^2-\omega^2)\,(c_t^2\xi_D^{-2}+c_t^2k^2-\omega^2)]=0 \qquad (7)$$

correspond to the direction vector $\mathbf{u}_0$ that lie in the plane of the vectors $\mathbf{k}$ and $\mathbf{n}$. Since Eq. (7) always has a positive determinant, the necessary and sufficient conditions for its having two different positive roots $\varkappa_1$ and $\varkappa_2$ (let $\varkappa_1<\varkappa_2$) is that both coefficients in the square brackets be positive. The quadratic equation

$$\varkappa^2=k^2-\omega^2/c_t^2$$

has a positive root $\varkappa_3$ and $\omega<c_t\,k$. The corresponding direction vector $\mathbf{u}_{03}$ is perpendicular to the vectors $\mathbf{n}$ and $\mathbf{k}$.

Substituting in (6) a linear combination of the three above-described solutions of Eq. (5) with arbitrary coefficients a_i, we can write down the dispersion equation of the surface spin waves as the condition for the compatibility of the obtained system of equations for these coefficients:

$$(c_l^2k^2-\omega^2)\,[2c_t^2\xi_D^{-2}(c_t^2+c_l^2\xi_0^2\xi_D^{-2}B_nB_t)+{}^1/_2(3c_t^2-c_l^2)\,(c_t^2+c_l^2)\,k^2-2c_t^2\omega^2]+[2c_l^4\xi_0^2\xi_D^{-4}B_nB_t+{}^1/_2(3c_t^2-c_l^2)\,(c_t^2+c_l^2)\,k^2-2c_t^2\omega^2]\,c_t^2\varkappa_1\varkappa_2+4c_t^2c_l^{10}\xi_0^2\xi_D^{-4}B_t^2\varkappa_1\varkappa_2(\varkappa_1+\varkappa_2)=0. \qquad (8)$$

The positive functions $\varkappa_1(\omega,k)$ and $\varkappa_2(\omega,k)$ in this equation were defined above.

By solving Eq. (8) it is possible to write down in explicit form the values of the coefficients *ai* in the sought linear combination, and obtain the explicit form of the distribution of the oscillations in the surface spin wave. We shall not do this here, but confine ourselves only to the remark that this combination does not contain a solution with a direction vector perpendicular to the vectors $\mathbf{n}$ and $\mathbf{k}$, but the two other solutions are contained in it. Thus, a surface spin wave exists only if Eq. (7) has two positive roots; in this case the vector $\mathbf{u}$ oscillates in the plane of the vectors $\mathbf{n}$ and $\mathbf{k}$.

To find the direction of the oscillation of the magnetization [which is expressed in terms of the time derivative of A^{ij} (Ref. 1)], it is necessary to change over from the vectors $\mathbf{u}$ of orbital space to the vectors of spin space with the aid of the matrix A^{ij}. This shows that the magnetization oscillations are elliptically polarized and take place in a plane perpendicular to the boundary and making an angle arccos $(-\frac{1}{4})$ with the wave vector.

ANALYSIS OF DISPERSION EQUATION OF SURFACE SPIN WAVES

In the analysis of the region $k\gg\xi_D^{-1}$, we can neglect the terms due to the dipole interaction, so that Eq. (8), after elimination of the irrational factors, reduces to the bicubic equation

$$\omega^6-2(c_t^2+c_l^2)\,k^2\omega^4+{}^1/_{16}(c_t^2+c_l^2)^2[24-(c_t^2+c_l^2)^2/c_t^2c_l^2]\,k^4\omega^2-{}^1/_{16}(c_t^2+c_l^2)^3[8-(c_t^2+c_l^2)^2/c_t^2c_l^2]\,k^6=0. \qquad (9)$$

In contrast to the analogous equation that arises in the problem of Rayleigh waves in an elastic solid (Ref. 3, §24), Eq. (9) can be resolved into rational factors. Only one of its three roots, namely

$$\omega^2={}^1/_2(c_t^2+c_l^2)\,[1-(c_t^2-c_l^2)/2c_tc_l]\,k^2,$$

corresponds to a real spectrum that lies below the spectra of the bulk modes (as should be the case for a surface mode), and furthermore only if c_t lies in the interval $c_l<c_t<(1+\sqrt{2})c_l$. From the fact that $c_t^2=2c_l^2$ for ^{3}He-*B* (Ref. 1) we can conclude that it has exactly one mode of surface spin waves that have a linear spectrum and a speed

$$c_s=[{}^3/_2(1-2^{-3/2})]^{1/2}c_l\approx0.985c_l.$$

A complete analysis of Eq. (8) with allowance for the terms corresponding to the dipole interaction) shows that at

$$k^2\leqslant k_0^2\approx\left(\frac{2c_l^2}{c_t^2-c_l^2}\right)^2B_t\frac{\xi_0}{\xi_D^3}\sim\frac{\xi_0}{\xi_D^3}$$

it has no purely real solutions such that Eq. (7) has two real positive roots. This means that there is no surface spin wave at $k\leqslant k_0$.

As $k\to k_0+0$ we have $\varkappa_1\to0$. This means that the "depth of penetration" of the surface spin wave into the interior of the liquid increases. At $k=k_0$ this penetration depth becomes infinite, and in place of a surface spin wave we obtain the usual bulk longitudinal spin wave with a wave vector parallel to the surface. In a small vicinity of the point k_0 (but of course at $k>k_0$) the surface of the surface spin wave takes (in first approximation) the form

$$\omega=c_lk-F(k-k_0)^2,\quad F\sim c_l\xi_0^3\xi_D^{-2}.$$

This curve is tangent at the point k_0 to the straight line $\omega=c_l\,k$ which comprises, for the given geometry, the spectrum of bulk longitudinal spin waves with a wave vector parallel to the surface.

The absence of surface spin waves at small k can be explained in the following manner. The presence in the energy [and hence also in the Lagrangian (4)] of a dipole orientational surface term should lead generally speaking to a gap-like spectrum at small k. At the same time, two of the three spin-wave bulk modes are gapless. Thus, at small k the spectrum of the surface spin wave should be located above the spectra of the two bulk spin-wave modes, but this is certainly impossible. The spectrum of an undamped surface wave can lie only above the spectrum of the bulk modes, otherwise its propagation is accompanied by emission of bulk waves and is subject to damping.

It must be noted that since the obtained frequency of the surface spin wave in the entire region $k_0 < k \ll \xi_0^{-1}$ of its existence is much higher than the frequency of a capillary-gravitational wave on a free ^{3}He-B surface, equal to

$$\omega=[gk+(\sigma/\rho)k^3]^{1/2} \tag{10}$$

for the same value of k, the result obtained above for ^{3}He-B bounded by a rigid noninteracting wall can be used also for the case of a free ^{3}He-B surface. Propagation of surface spin waves leaves the surface practically plane (the rotation angles of the normal vector **n** will be much smaller than the rotation angles of the matrix A^{ij}) because the mechanical motions are relatively slow.

The presence of a surface dipole interaction, it is not only the spin waves that lead to certain (albeit negligibly small) oscillations of the free surface, but also on the contrary, the mechanical motions accompanied by the surface oscillations inevitably give rise to oscillations of A^{ij}.

A combined analysis of the equations of hydrodynamics and of the spin dynamics of ^{3}He-B shows that the surface dipole interaction splits the spectrum of the capillary-gravitational waves into two branches that correspond to different polarizations of the accompanying spin motion. The frequency of each of these branches, however, differs from the frequency given by (10) by not more than $b^{ij}/\sigma \sim 10^{-9}$.

SPIN WAVES IN A THIN LAYER

Besides the surface spin waves, one can observe also two-dimensional spin excitations such as spin waves in a planar thin layer.

When dealing with distributed oscillations, the problem of oscillations in a thin layer (plate) is treated separately. This problem is regarded as relatively simple, since it can be solved without determining the explicit distribution of the oscillations over the layer thickness. By calling the layer thin in this case, one implies that such an approach, which does not take its finite thickness into account, is applicable to the layer. For spin waves in ^{3}He-B this method can be used only in the limiting case $\xi_D^{-1} \ll k \ll L^{-1}$ (L is the layer thickness), when one can neglect besides the finite layer thickness also effects connected with dipole interaction. The analysis that follows is restricted to just this last case.

We use a modification of the customary method (see, e.g., Ref. 3, §25) of solving the problem of elastic oscillations in a thin plate. We represent the vectors $\partial/\partial x^k \equiv \nabla^k$ and **u** in the form of the sums $\nabla_n + \nabla_t$ and $\mathbf{u}_n + \mathbf{u}_t$ of vectors normal (n) and tangent (t) to the plane of the thin layer. We can then separate in each of Eqs. (1) and (2), which must be used to solve this problem, the parts normal and tangent to the thin layer, and treat them subsequently as independent equations. Application of the operators ∇_n and ∇_t to the equations obtained from (2) in this manner yields relations with which the equations obtained from (1) are reduced to a form that does not contain ∇_n:

$$\partial \mathbf{u}_n^2/\partial t^2=c_{sn}^2\Delta_t\mathbf{u}_n, \tag{11}$$

$$\partial^2\mathbf{u}_t/\partial t^2=c_{st}^2\Delta_t\mathbf{u}_t+(c_{sl}^2-c_{st}^2)\nabla_t(\nabla_t\mathbf{u}_t), \tag{12}$$

where

$$\begin{gathered}\Delta_t=\nabla_t^2,\quad c_{sn}^2=(3c_t^2-c_l^2)(c_t^2+c_l^2)/4c_t^2={}^{15}/_8c_l^2,\\ c_{st}^2=c_t^2=2c_l^2,\quad c_{sl}^2=(3c_l^2-c_t^2)(c_l^2+c_t^2)/4c_l^2={}^3/_4c_l^2.\end{gathered} \tag{13}$$

Equation (11) describes a spin wave polarized perpendicular to the plane of the layer, while Eq. (12) describes one polarized in the plane of the layer; the latter, obviously, breaks up into two individual branches, longitudinal and transverse. All three waves have a linear spectrum, and their velocities are given by relations (13). We recall that here, as before, wave polarization is taken to mean the direction of the oscillations of the vector. The direction of the magnetization oscillations can be obtained from this direction by rotation through an angle arccos $(-\frac{1}{4})$ around **n**.

Since the remark made at the end of the preceding section is equally applicable to the present situation, the results are valid whether the film is bounded on any side by a rigid noninteracting wall or by the free surface of ^{3}He-B.

Thus, despite the similar mathematical formulation, for spin waves in a thin ^{3}He-B layer we have a situation somewhat different from that for elastic waves in thin plates, where only two waves have a linear spectrum, and the wave polarized perpendicular to the plane of the plate (the flexure wave) has a spectrum of the form $\omega \propto Lk^2$.

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