Possible splitting of a phase transition in a 2*D XY* model

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A change in the type of interaction in a 2*D XY* model can change the nature of a phase transition and can even split it in two. One of the resulting transitions may be an Ising transition, for example. An effect of this type may occur in thin films of superfluid ^{3}He-*A*.

The Hamiltonian of the *XY* model, which has a symmetry $O(2)$ is

$$H = \sum_{(jl)} V(\varphi_j - \varphi_l); \qquad V(\Delta\varphi) \equiv - V_0 \cos(\Delta\varphi), \tag{1}$$

where the sum is over pairs of nearest neighbors in the plane lattice, and the variables φ_j are defined on the ring $-\pi \leqslant \varphi_j \leqslant \pi$. In model (1), a phase transition occurs between phases with power-law and exponential decays of the correlation function[1] $\langle \exp i(\varphi_j - \varphi_l) \rangle$. This transition involves the dissociation of a pair of vortices and is an infinite-order transition according to the Ehrenfest classification.[1]

Let us consider a modification of the *XY* model in which $V(\Delta\varphi)$ is an even periodic function of $\Delta\varphi$ which has at $\Delta\varphi = \pi$ yet another minimum, of nearly the same depth as the absolute minimum at $\Delta\varphi = 0$ (Fig. 1). In this case the extrema of the Hamiltonian that contribute to the partition function are not only vortices and vortex pairs but also solitons: line singularities on which φ changes by π. The solitons may be closed, and they may terminate at vortices with a circulation of $\pm\pi$ (half-vortices).

The energy of a soliton per unit length is equal to the difference (V_1) between the depths of the minima of the functions $V(\Delta\varphi)$. If $V_1 \sim T$, the free energy per unit length of the soliton vanishes. At a higher temperature the correlation function $\langle \exp i(\varphi_j - \varphi_l) \rangle$ falls off exponentially, since the remote points j and l are separated by a large number of solitons, on each of which φ has a discontinuity of π. The correlation function $\langle \exp[2i(\varphi_j - \varphi_l)] \rangle$ nevertheless falls off in a power-law fashion, since the flexural stiffness is preserved in the system at $T \ll V_2$. At $V_1 \ll T \ll V_2$ the system is accordingly in an intermediate phase, in which the spontaneously broken symmetry with respect to the 2*D* rotation group is partially restored (for rotations that differ by an angle of π). The transition between the intermediate and low-temperature phases involves a breaking of the group Z_2, although it is not possible to single out a variable of the Ising type with respect to which an ordering occurs because of the absence of a strict long-range order in terms of φ.

Although the solitons are not topologically nonremovable singularities (since they may terminate at half-vortices), at $V_1 \ll T \ll V_2$ their free ends are bound by the strong logarithmic interaction into small pairs that lie far from each other. Over scale dimensions greater than the average dimension of a pair, the presence of these "droplets" in solitons could hardly be important.

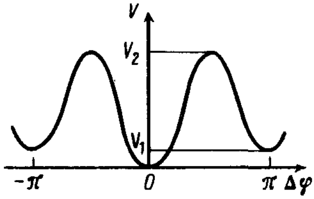

FIG. 1.

To establish the existence of the intermediate phase and of the Ising transition more rigorously, we consider a model involving this modification of the XY model by a dual transformation.[2] This is an SOS model with the Hamiltonian

$$\widetilde{H} = \sum_{(jl)} \widetilde{V}(n_j - n_l), \tag{2}$$

where the sum is over pairs of nearest neighbors on the dual lattice, and the interaction $\widetilde{V}(n_j - n_l)$ of the integer variables n_j depends strongly on the parity of $n_j - n_l$. Let us assume, for example,

$$\widetilde{V}(n_j - n_l) = \frac{J}{2}(n_j - n_l)^2 - \frac{K}{2}\sigma_j\sigma_l; \qquad \sigma_j \equiv \exp i\pi n_j$$

The case $V_1 \ll T \ll V_2$ corresponds to $J \ll 1 \ll K$ (we assume that the temperature is incorporated in the definition of $\widetilde{H}$). At $K = \infty$, the variables n_j are either all odd or all even. If $J \ll 1$, then the system is in a "rough" state; i.e., the square of the width of the surface diverges, and the free energy per unit length of a step (of height 2) vanishes. For $1 \ll K < \infty$, domain walls may form between these two rough "vacuums." These domain walls have a large energy per unit length K and thus a finite free energy (an upper limit on the order of unity can be estimated for the entropy). There are only small "islands" of one "vacuum" in the other, and the order in terms of the Ising variable σ is preserved.

A finite free energy of a step of unit height in terms of our original XY model means an exponential decay of the correlation function[3] $\langle \exp i(\varphi_j \varphi_l)\rangle$. The vanishing of the free energy of a step of height 2 means a power-law decay of the correlation function $\langle \exp[2i(\varphi_j - \varphi_l)]\rangle$. For $J \ll 1 \ll K$, the XY model, which is the dual of (2), is thus in the intermediate phase described above.

Retaining the σ_j as independent variables, we can transform the partition function of model (2) into the partition function of a $2D$ Coulomb gas by the standard procedure[4]:

$$Z = \sum_{\sigma = \pm 1} \sum_{m=-\infty}^{\infty} \exp\left\{ -\frac{K}{2} \sum_{(jj')} \sigma_j \sigma_{j'} - \frac{2\pi^2}{J} \sum_{j,l} \frac{m_j}{2} G_{jl} \frac{m_l}{2} + i\pi \sum_j \sigma_j \frac{m_j}{2} \right\}, \tag{3}$$

where the interaction G_{jl} of the integer charges m_j (which are vortices with a circulation πm_j) is logarithmic at long range. For $J \ll 1 \ll K$, the charges are bound in pairs, and the Ising variables σ_j are ordered. What are the consequences of the interaction of σ and m?

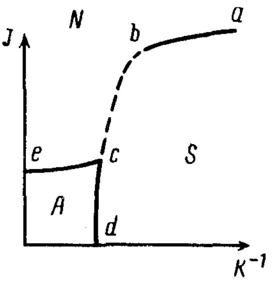

FIG. 2.

For $J \ll 1$ we can carry out the summation over m_j in (3), assuming that these charges are bound in neutral pairs that are far from each other, and whose interaction with each other simply causes a renormalization of J. After the summation is carried out, we find an additional ferromagnetic interaction of the variables σ, which decays at long range in proportion to $r^{-\pi/2J_R}$. If the half-vortices are bound in pairs, then we have $\pi/2J_R \geqslant 4$, and this additional interaction cannot lead to a change in the nature of the Ising transition.

If we consider the inverse effect of the Ising variables on the vortices, we see that a term $-\ln\langle\sigma_j\sigma_l\rangle$ is added to the nucleating interaction of half-vortices ($m = \pm 1$). If $\langle\sigma_j\rangle \neq 0$, the situation is equivalent to simply a decrease in the activity of the half-vortices, while if $\langle\sigma_j\rangle = 0$ an interaction proportional to the distance between half-vortices arises between these half-vortices (i.e., the energy per unit length of a soliton is finite).

Figure 2 is a schematic phase diagram in the K^{-1}, J plane. Here S denotes the ordered phase of the XY model, N is the disordered phase, and A is the intermediate phase. Pairs of ordinary vortices dissociate on the line ab, the free energy of a soliton vanishes on the line bd, and pairs of half-vortices dissociate on the line ce. In region bc, where the free energy of the soliton vanishes, the logarithmic interaction between the half-vortices is too weak to bind them in pairs. Consequently, the transition to the disordered state in this case must occur in a way different from that in region ab (i.e., either with different indices or by means of the first-order transition).

An example of a physical system which has properties similar to those of this model is ^{3}He-A film in a strong magnetic field. If the field is strictly perpendicular to the film, the interaction of the half-vortices (which in this case are simultaneously disclinations with Frank indices $\pm 1/2$) is purely logarithmic.[5] If the field makes a small angle with the perpendicular to the film, a term which is attributable to the appearance of a soliton and which is linear in the distance arises. As the temperature and the inclination of the field are varied, we should observe a phase diagram like that shown in Fig. 2.

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