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# Phase Transitions in Two-Dimensional Uniformly Frustrated *XY* Models. II. General Scheme

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For two-dimensional uniformly frustrated $XY$ models the group of symmetry spontaneously broken in the ground state is a cross product of the group of two-dimensional rotations by some discrete group of finite order. Different possibilities of phase transitions in such systems are investigated. The transition to the Coulomb gas with noninteger charges is widely used when analyzing the properties of relevant topological excitations. The number of these excitations includes not only domain walls and traditional (integer) vortices, but also vortices with a fractional number of circulation quanta which are to be localized at bends and intersections of domain walls. The types of possible phase transitions prove to be dependent on their relative sequence: in the case the vanishing of domain wall free energy occurs earlier (at increasing temperature) than the dissociation of pairs of ordinary vortices, the second phase transition is to be associated with dissociation of pairs of fractional vortices. The general statements are illustrated with a number of examples.

**KEY WORDS:** Two-dimensional systems; phase transitions; frustrated $XY$ models; topological excitations; fractional vortices; Josephson junctions; superfluid $^3$He–A thin films.

## 1. INTRODUCTION

This paper presents the second part of the study of phase transitions possible in two-dimensional uniformly frustrated $XY$ models. The first part, which will be hereafter referred to as part I, is published as a separate paper.[1] The models considered can be described by the Hamiltonian

$$H = \sum_{(\mathbf{rr}')} V(\varphi_{\mathbf{r}} - \varphi_{\mathbf{r}'} - \psi_{\mathbf{rr}'}) \tag{1.1}$$

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Here the phase variables $\varphi_{\mathbf{r}}$ are defined on the sites of a regular two-dimensional lattice and the summation is performed over pairs of nearest neighbors. $V(\Delta\varphi)$ is an even periodic function of its argument $\Delta\varphi$ and has its minimum at $\Delta\varphi = 0$. The constants $\psi_{\mathbf{r}\mathbf{r}'}$ ($\psi_{\mathbf{r}\mathbf{r}'} \equiv -\psi_{\mathbf{r}'\mathbf{r}}$) are defined on lattice bonds in such manner that their sum along the perimeter of each elementary plaquette assumes the same value $2\pi f$.

The Hamiltonian (1.1) with $V(\Delta\varphi) = -J\cos(\Delta\varphi)$ can be applied for the description of regular arrays of Josephson junctions in a perpendicular magnetic field (see, e.g., Refs. (2–5)). In part I a model with the triangular lattice and $f = \frac{1}{2}$ has been considered in detail. This so-called AF $XY$(t) model can also describe a planar antiferromagnet.

If the interaction function $V(\Delta\varphi)$ is chosen in the Berezinskii–Villain form[6,7]

$$\exp\left[-\frac{V(\Delta\varphi)}{T}\right] = \sum_{p=-\infty}^{\infty} \exp\left[-\frac{J}{T}(\Delta\varphi - 2\pi p)^2\right] \tag{1.2}$$

the partition function of the model (1.1) can be transformed to the partition function of two-dimensional Coulomb gas with charges assuming noninteger values shifted with respect to the integer values by $-f$.[8] The small variation of the interaction function $V(\Delta\varphi)$ does not affect the classification of topological defects. Therefore, when analyzing the latter we shall widely use the Coulomb gas representation, the rigorous transformation to which is possible only in the case of Berezinskii–Villain interaction (1.2).

In Section 2 phase transitions possible in the model (1.1) with $f = \frac{1}{2}$ and the square lattice are studied. The models considered in part I and Section 2 of this paper are equivalent to the lattice Coulomb gas with half-integer charges. The possibility of experimental realization of such Coulomb gas in superfluid $^3$He–A thin films is discussed in Section 3. In Section 4 the general scheme of phase transitions possible in the models (1.1) with rational $f$ is presented. The types of phase transitions associated with breaking of the discrete and continuous symmetries prove to be dependent on the relationship between the transition temperatures. The general statements of Section 4 are illustrated in Section 5 on the example of the model with the honeycomb lattice and $f = \frac{1}{3}$. In Section 6 some examples of the models with the additional (accidental) degeneracy of the ground state are considered. In such cases the structures of the ordered states at low temperatures can be determined only when taking into account the free energy of spin waves. In Section 7 the dependence of transition temperatures on $f$ is briefly discussed.

## 2. VILLAIN'S ODD MODEL (SQUARE LATTICE, $f=\frac{1}{2}$)

The AF $XY$(t) model considered in part I is equivalent to Coulomb gas with half-integer charges defined on a honeycomb lattice. We will begin our acquaintance with other uniformly frustrated $XY$ models by studying a model equivalent to the Coulomb gas with half-integer charges on a square lattice. Such a model can be interpreted as a model of a magnet possessing both ferromagnetic and antiferromagnetic bonds (so-called Villain's odd model,[9] Fig. 1).

Properties of this model prove to be rather close to those of the AF $XY$(t) model. In particular, the symmetry group which is broken in the ground state is the same: $U(1)\times Z_2$.[9] The twofold discrete degeneracy manifests itself in the Coulomb gas representation as the twofold degeneracy of the ground state in which charges $+\frac{1}{2}$ and $-\frac{1}{2}$ are situated in check order.[2]

As in the case of the AF $XY$(t) model the relevant excitations are integer excessive charges (positive or negative) and neutral domain walls. In the case of a square lattice, fractional charges prove to be localized at every bend of the domain wall. Using the simple algorithm described in part I (Section 3), one can easily verify that at the points $a$, $b$, $c$, and $e$ of the domain wall (Fig. 2) the charges $+\frac{1}{4}$, $-\frac{1}{4}$, $-\frac{1}{4}$, and $+\frac{1}{4}$ are localized, respectively. At the intersection of two domain walls the half-integer charge is localized (Fig. 2, point $d$).

The inevitable appearence of the fractional charge at every bend of the domain wall is due to a specific form of the "correspondence rules" analogous to those given by (2.1) of part I. If one fixes the state at one side

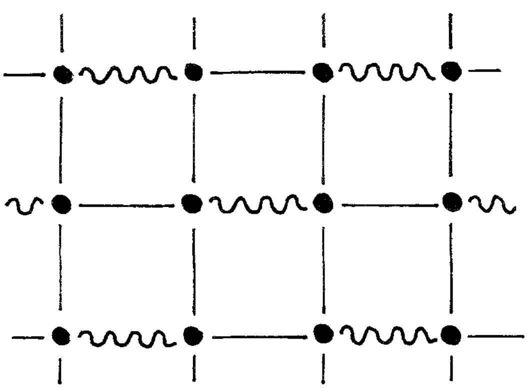

Fig. 1. Villain's odd model[9]: a planar magnet with ferromagnetic and antiferromagnetic bonds (shown by straight and by wiggly lines, respectively). Each plaquette contains an odd number of antiferromagnetic bonds.

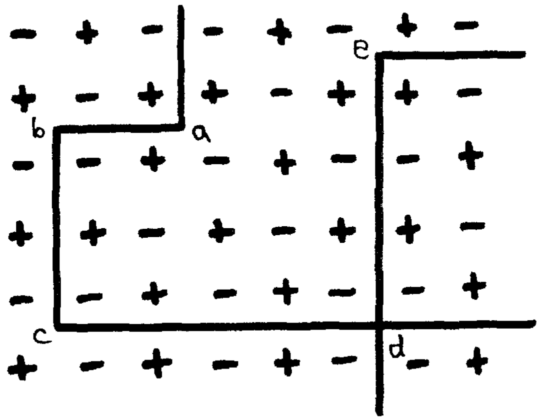


Fig. 2. Domain walls in Coulomb gas of half-integer charges on a square lattice.

of the domain wall, the choice of a state at the other side yielding true Hamiltonian extremum becomes dependent not only on the position of the domain wall but also on its orientation. The states obtained if a horizontal or a vertical domain wall is crossed can be transformed one into another by the rotation by the angle of 90°. Thus, if a domain wall has a 90° bend, a vortex with a quarter of a circulation quantum must be localized at this bend to make the configuration of the field $\varphi$ a true Hamiltonian extremum.

The "generalized" phase diagram assumes roughly the same form as the phase diagram of the generalized AF $XY$(t) model, depicted in Fig. 4 of part I. At the phase transition on the line *de* in this case the octets of points of the order parameter degeneracy space merge into single points, each octet consisting of two quarters belonging to different circumferences. On the line *df* the dissociation of "quarter vortices" takes place.

The possibility of a "domain wall" transition occurring at the temperature low enough for fractional vortices to be bound in pairs in the considered case is relatively narrow due to the absence of such domain wall fluctuations that would not induce the appearance of fractional vortices. If one considers the model (1.1) with an interaction function $V(\Delta\varphi)$ having such a form that the domain wall energy (per unit length) $J_W$ is much less than a prelogarithmic factor $J_V$ in the long ranged vortex–vortex interaction, the typical form of the thermally activated defects (closed domain walls) will be such as is shown in Fig. 3. With increasing temperature up to $J_V/\ln(J_V/J_W)$ the mean distance between these defects becomes the same order as their size. Their merging with further increasing temperature can lead to appearence of infinite domain walls which corresponds to vanishing of the domain wall free energy. In contrast to the AF $XY$(t) model the type

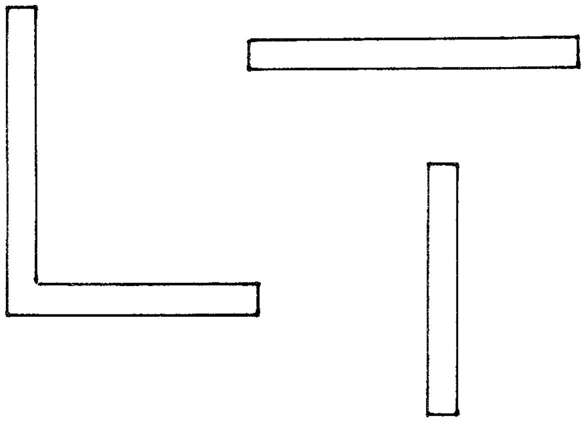

Fig. 3. Typical form of thermally activated closed domain walls for Villain's odd model on a square lattice.

of the transition on the line *bf* cannot be determined by means of studying some discrete symmetry model with the nearest neighbor interaction, because in the initial model *all* bends and intersections of the domain walls interact logarithmically.

The Monte Carlo stimulation of the model (1.1) with $f = \frac{1}{2}$ on a square lattice[2] showed the logarithmical divergence of the specific heat, characteristic of the Ising transition. The accuracy of the obtained results did not let the authors make definite conclusions about the relative order of the Berezinskii–Kosterlitz–Thouless (BKT) transition, associated with the breaking of the continuous symmetry, and of the Ising-type transition.

Our analysis of relevant topological excitations enables us to assert unambiguously that if the "domain wall" transition is of the Ising type, then the BKT transition must occur at lower temperatures. In the case of the inverse relative order of the transitions the domain wall transition would correspond to some $Z_4 \times Z_2$ model with long-ranged interaction and the BKT transition would be associated with dissociation of quarter-vortex pairs and would occur when the helicity modulus is equal to $(32/\pi)T$.

In the remaining part of this section some general properties of the defects interaction that are widely used in part I and in this paper will be illustrated on the example of the Villain's odd model with a square lattice.

The partition function of the considered model with the accuracy of a nonsingular factor can be presented as the partition function of the Coulomb gas with half-integer charges $m(\mathbf{R})$, defined on the sites of the dual (also square) lattice (see Ref. (10)). The corresponding Hamiltonian has the form:

$$H = \sum_{\mathbf{R},\mathbf{R}'} m(\mathbf{R})\, G(\mathbf{R} - \mathbf{R}')\, m(\mathbf{R}') \tag{2.1}$$

with interaction $G(\mathbf{R}-\mathbf{R}')$ given by

$$G(\mathbf{R}-\mathbf{R}') = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \tilde{G}(\mathbf{k}) \exp i\mathbf{k}(\mathbf{R}-\mathbf{R}') \tag{2.2}$$

where

$$\tilde{G}(\mathbf{k}) = (2\pi)^2 J/[4\sin^2(k_x/2) + 4\sin^2(k_y/2)] \tag{2.3}$$

(the lattice unit is used like a length unit).

The configuration of charges in the ground state is explicitly given by:

$$m^{(0)}(\mathbf{R}) = \pm\tfrac{1}{2} \exp i\mathbf{k}_*\mathbf{R}; \qquad \mathbf{k}_* = (\pi, \pi) \tag{2.4}$$

Different signs in (2.4) correspond to two possible ground states. Substituting (2.4) into (2.1) one can easily find the ground state energy:

$$E^{(0)} = \tfrac{1}{8} N \tilde{G}(\mathbf{k}_*) \tag{2.5}$$

where $N$ is the total number of sites of the dual lattice (infinite in thermodynamic limit).

Consider the excessive charges on the background of the ground state. Let the configuration of charges be

$$m(\mathbf{R}) = m^{(0)}(\mathbf{R}) + m' \cdot \delta_{\mathbf{R}\mathbf{R}_1} - m' \cdot \delta_{\mathbf{R}\mathbf{R}_2} \tag{2.6}$$

i.e., there is an excessive charge $m'$ at the point $\mathbf{R}_1$, and an excessive charge $-m'$ at the point $\mathbf{R}_2$. The energy of the configuration (2.6) is given by

$$E_{2V} = E^{(0)} + (m')^2[G(\mathbf{0}) - G(\mathbf{R}_1 - \mathbf{R}_2)] + \tfrac{1}{2}\tilde{G}(\mathbf{k}_*)[m' \cdot m^{(0)}(\mathbf{R}_1) - m' \cdot m^{(0)}(\mathbf{R}_2)] \tag{2.7}$$

Equations (2.5) and (2.7) are valid for an arbitrary form of the interaction function (2.2) (retaining the same form of the ground state).

It becomes evident from (2.7) that the regular background of half-integer charges does not change the strength of the logarithmic interaction of excessive charges. Only their "core" energy is affected and is varied by a value proportional to $m'(\mathbf{R})\, m^{(0)}(\mathbf{R})$. The interaction entering the Hamiltonian (2.1) being quadratic in $m(\mathbf{R})$, these statements remain valid for an arbitrary number of nonzero excessive charges $m'(\mathbf{R})$. Considering the structure of (2.7) one can further extend this statement for an arbitrary regular background structure (with $f$ not necessarily equal to $\tfrac{1}{2}$).

Let us calculate then the domain walls interaction energy. Two domain walls separated by a distance $L$ can be represented, for instance, by the configuration of charges:

$$m(\mathbf{R}) = \begin{cases} m^{(0)}(\mathbf{R}), & x \leqslant 0 \quad \text{or} \quad x \geqslant L+1 \\ -m^{(0)}(\mathbf{R}), & 1 \leqslant x \leqslant L \end{cases} \tag{2.8}$$

where $\mathbf{R} = (x, y)$. Substituting (2.8) into (2.1) we can find that the respective energy is given by

$$E_{2W} = E^{(0)} + N_y \frac{(2\pi)^2 Ja^2}{(1-a)(1+a)^3} [1 - (-a)^L] \tag{2.9}$$

Here $N_y$ is the size of the system in the direction of the $y$ axis (parallel to domain walls) and $a = 3 - \sqrt{8} \approx 0.1716$. Equation (2.9) was obtained for a particular form of the interaction function given by (2.3).

Thus we can conclude that domain walls have finite energy per unit length and their interaction decays exponentially with a distance. The latter circumstance allows us to neglect this interaction in a qualitative analysis of possible phase transitions.

## 3. COULOMB GAS WITH HALF-INTEGER CHARGES AND SUPERFLUID $^3$He–A THIN FILMS

The models considered in part I and Section 2 of this paper allow for a transformation to the lattice Coulomb gas with half-integer charges. It seems worth mentioning that such Coulomb gas can be experimentally realized in thin films of superfluid $^3$He–A.

The continuously variable part of the order parameter for $^3$He–A thin film can be written as

$$\Psi = \mathbf{d}e^{i\chi} \tag{3.1}$$

where $\mathbf{d}$ is a unit vector. When applying the magnetic field exceeding $H_c \approx 50$ G perpendicular to the film the effective anisotropy field for vector $\mathbf{d}$ becomes the "light plane" type,[11] so the vector $\mathbf{d}$ can be treated as planar.

The order parameter degeneracy space in this case is $(S^1 \times S^1)/Z_2$. The factorization by $Z_2$ is caused by pairs $\mathbf{d}$, $\chi$ and $-\mathbf{d}$, $\chi + \pi$ giving the same value of the order parameter (3.1).

The simplest point singularities for the order parameter (3.1) (where the vector $\mathbf{d}$ is treated as planar) are vortices of the phase $\chi$ and disclinations of the vector $\mathbf{d}$. Less trivial point singularities are so-called exotic vortices.[12] Each one of those is a combination of a vortex with a half-integer number of circulation quanta and of a disclination with the half-integer Frank index.[12]

When the film is rotated the formation of the lattice of these exotic vortices becomes possible.[12] The sign of the superfluid velocity circulation evidently will be the same for all the vortices of the lattice (it is determined by the direction of the rotation). The "disclination" degrees of freedom, however, remain at our disposal: the Frank indices of the exotic vortices can acquire arbitrary half-integer values and therefore form a lattice gas of half-integer charges with logarithmic interaction. So far it is not yet clear what particular kind of the lattice will be formed.

In the experimental situation the melting of the exotic vortex lattice may take place at a temperature not high enough for phase transitions in the Coulomb gas of Frank indices to occur. In this case the observation of these transitions would be possible only if one stabilizes the structure of the vortex lattice by means of the regular array of pinning centers.

## 4. GENERAL SCHEME

The situation we encounter when studying the AF $XY$(t) model and Villain's odd model—two phase transitions the types of which are dependent on their relative order—proves to be typical of the whole class of the two-dimensional uniformly frustrated $XY$ models. In the case of positive rational $f$ smaller than $\frac{1}{2}$ the ground state (in the Coulomb gas representation) is a regular structure on a dual lattice consisting of positive charges $1-f$ and negative charges $-f$.[3] If one denotes the discrete symmetry group associated with the degeneracy of this state as $G_P$ ($P$ is the order of the group), then the whole group spontaneously broken in the ground state is $U(1)\times G_P$.

As in the simplest cases considered previously, two different phase transitions can occur: (1) melting of the regular structure formed by charges $1-f$ on the background of charges $-f$ (vanishing of the mean charge of each sublattice); and (2) appearence of free excessive charges associated with the screening of their logarithmic interaction. We would like to recall that in the mean field analysis of such systems[3–5] it has been supposed that only one phase transition takes place.

If the dissociation of pairs of excessive charges occurs first (with increasing temperature), then it is the traditional BKT transition and the second transition is associated with group $G_P$. In the other case when melting of the "ionic-crystallic" structure of charges occurs at lower temperature, it is associated with restoring of the group $Z_Q\times G_P$, where $1/Q$ is the minimal fractional charge out of those that happen to be localized at defects in the regular domain walls. The examples considered above show that $1/Q$ does not have to coincide with $f$. In this case the second transition is associated with dissociation of pairs of charges $\pm 1/Q$. The helicity modulus is equal to $(2Q^2/\pi)T$ at the transition point. The third opportunity (which is to be realized at intermediate values of the parameters) is simultaneous restoring of the whole group $U(1) \times G_P$. A question of the type of this transition seems rather intriguing.

The reason for our always mentioning the values of the helicity modulus $\gamma$ at the transition point is that in numerical simulations the temperature of the BKT transition is often determined as a temperature where $\gamma$ is equal to its universal value at the transition point $(2/\pi)T$ (see, e.g., Refs. (2, 5, 13)). In case of BKT transition in a system of fractional vortices the universal value of $\gamma(T_{\mathrm{BKT}})$ changes.

Shih and Stroud[5] interpreted the results of their Monte Carlo simulation as evidence for the existence of two phase transitions in the model (1.1) with $f = \frac{1}{4}$ and a triangular lattice, the second (with increasing temperature) of these transitions being the ordinary BKT transition. The analysis presented above allows us to decline this assumption: this sequence of transitions is impossible. In this context the results of Shih and Stroud's numerical simulation[5] give evidence rather to the existence of only one phase transition or reveal that the second transition is associated with dissociation of pairs of fractional vortices and occurs at temperature where $j$ is equal to $(2Q^2/\pi)T$ $(Q > 1)$ and not $(2/\pi)T$.

In the general case the possibility of splitting of a phase transition associated with the group $G_P$ or $Z_Q \times G_P$ into two or more different transitions cannot be excluded. It seems worth mentioning that even in the $z_4$-model (Ashkin–Teller model) splitting of the transition is possible (see, e.g., Refs. (14–16)).

In the next section a simple example illustrating our general scheme will be presented.

## 5. HONEYCOMB LATTICE, $f = \frac{1}{3}$

In the case of the model (1.1) on honeycomb lattice the corresponding Coulomb gas is defined on a triangular lattice. For $f = \frac{1}{3}$ the ground state of this Coulomb gas is a commensurate crystal $\sqrt{3} \times \sqrt{3}$ of charges $+\frac{2}{3}$ on the background of charges $-\frac{1}{3}$. This state is evidently three-fold degenerate, so the whole group broken in the ground state is $U(1) \times Z_3$.

In this model domain walls separating different ground states can be treated as consisting of links of a length $\sqrt{3}$ (in lattice units), see Fig. 4. A simple calculation analogous to that of Section 3 of I shows that at each link a charge $\pm\frac{1}{3}$ is localized. For neighboring links forming an angle of

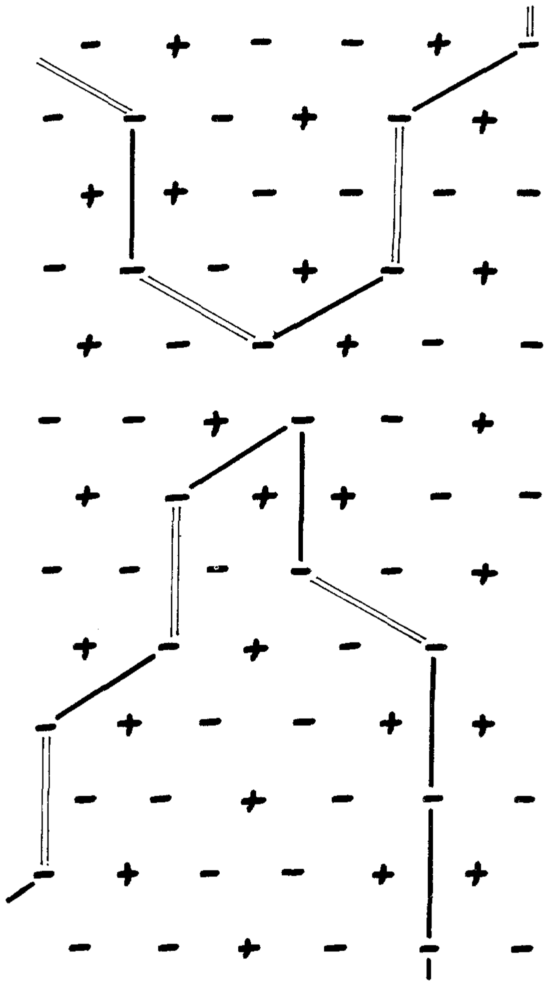

Fig. 4. Domain walls in Coulomb gas corresponding to the model (1.1) with a honeycomb lattice and $f=\frac{1}{3}$. Charges $+\frac{2}{3}$ and $-\frac{1}{3}$ are denoted by pluses and minuses, respectively. The domain wall links with positive and negative charges are shown by black and white lines.

120° the charges are opposite in sign, and in the case of angles 60° or 180° they are of the same sign.

Thus for the domain wall consisting of links forming 120° joints the charges of neighboring links compensate each other. If somewhere a 60° or 180° joint occurs, a charge $\pm\frac{1}{3}$ is localized on it. The situation proves to be quite analogous to that encountered when studying the AF $XY$(t) model (see part I). Note that although in the initial formulation positive and negative charges have had different values, this asymmetry is lost for charges localized on domain wall links.

The carried out analysis makes it possible to enumerate all possible sequences of phase transitions for the considered model:

1. Ordinary BKT transition (dissociation of pairs of integer vortices) and then (with further increasing temperature) melting of the $\sqrt{3} \times \sqrt{3}$ structure, characterized by the same critical exponents as a phase transition in the three-state Potts model.[17]
2. Restoring of the group $Z_3 \times Z_3$ and then the dissociation of pairs of fractional vortices with one-third of the circulation quantum.
3. Simultaneous restoring of the whole group $U(1) \times Z_3$.

With varying interaction function $V(\Delta\varphi)$ all these three opportunities should be realized.

## 6. SOME OTHER EXAMPLES

In some cases there arises a complication not included in the general scheme of Section 4. Consider Villain's odd model (i.e., the model (1.1) with $f = \frac{1}{2}$) on a honeycomb lattice.

Figure 5a schematically gives one of the possible regular structures minimizing the Hamiltonian. Shaded hexagons correspond to charges (helicities) $+\frac{1}{2}$ and light, to charges (helicities) $-\frac{1}{2}$. Bold lines are used to show the bonds for which the absolute value of the gauge-invariant difference $(\Delta\varphi)_{\mathbf{rr}'} \equiv \varphi_{\mathbf{r}} - \varphi_{\mathbf{r}'} - \psi_{\mathbf{rr}'}$ is equal to $\pi/4$. For other bonds, $(\Delta\varphi)_{\mathbf{rr}'}$ equals zero. Summation of $(\Delta\varphi)_{\mathbf{rr}'}$ along the perimeter of each hexagon gives $\pm\pi$.

On the background of this ground state (Fig. 5a) a domain wall with zero energy can be created (Fig. 5b). A regular set of such domain walls (Fig. 5c) is another type of the ground state having a regular structure. The additional (not related to symmetry) degeneracy described here is not connected with the particular form of the interaction function entering the Hamiltonian (1.1).

At finite temperatures this additional degeneracy may be removed due to thermal fluctuations. In the case of the Berezinskii–Villain interaction (1.2) the term giving the energy of spin waves can be separated from the other part of the Hamiltonian, so the free energy of spin waves is identical for all configurations of charges. The difference of free energies of the states depicted in Figs. 5a and 5c is in this case caused only by different contributions from thermally activated bound pairs of excessive charges. These contributions are different due to dependence of the excessive charge core energy on the relation between its sign and the sign of the "background"

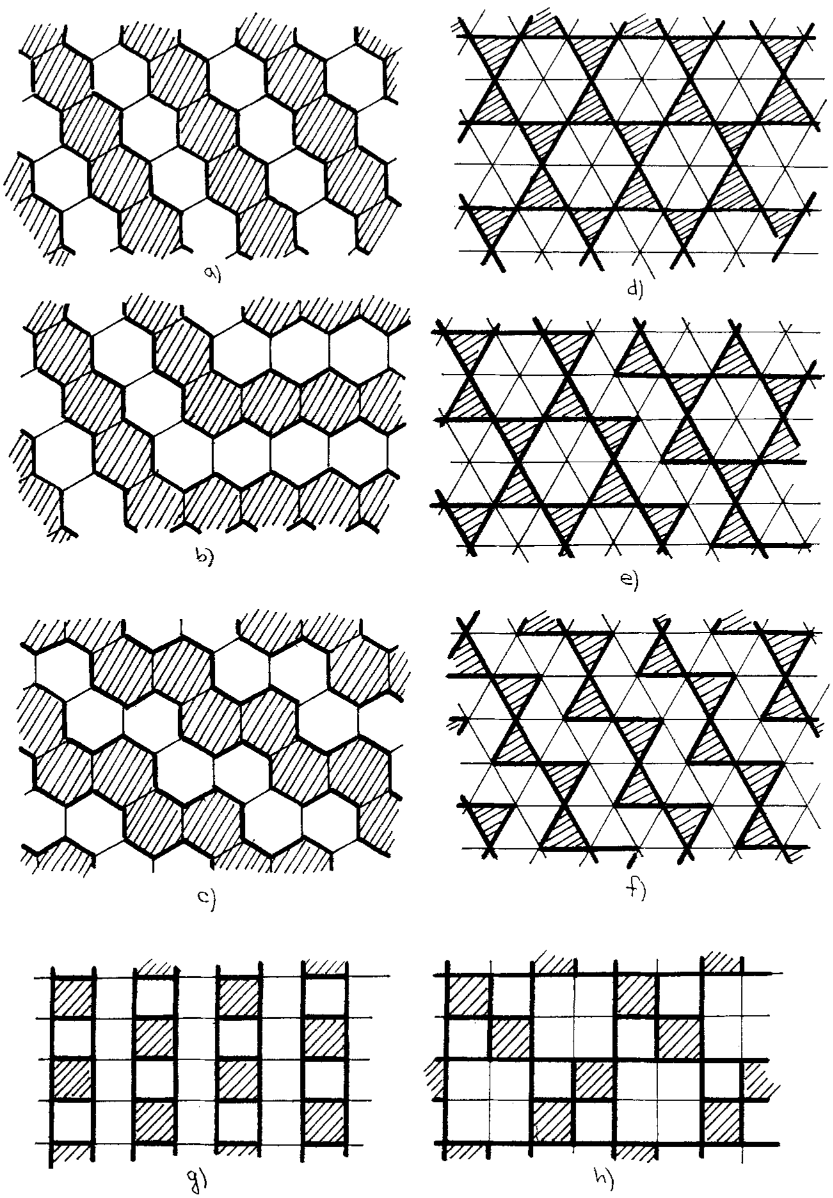


Fig. 5. Some examples of the ground states for uniformly-frustrated $XY$ models: $(a, b, c)$ a honeycomb lattice, $f=\frac{1}{2}$; $(d, e, f)$ a triangular lattice, $f=\frac{1}{4}$; $(g, h)$ a square lattice, $f=\frac{1}{4}$. Shaded cells have helicities $1-f$, light ones $-f$. The bonds for which $(\Delta\varphi)_{rr'}=0$ are shown by thin lines.

charge (see Section 2). For low temperatures the difference between the free energies of two considered states is exponentially small in $J/T$.

For the interaction function $V(\Delta\varphi)$ distinct from the one given by Eq. (1.2) (in particular for the physically interesting case $V(\Delta\varphi) = -J\cos(\Delta\varphi)$) the difference between the free energies should emerge even at the calculation of the spin waves' free energy. In harmonic approximation, however, the total free energy of spin waves proves to be equal for both regular structures considered. Thus the question about the true nature of the ordered state at low temperatures can be answered only if anharmonicities are taken into account. One can expect that it corresponds to one of the states depicted in Figs. 5a and 5c, the broken symmetry being a cross product of a group of two-dimensional rotations and of a discrete group of the sixth order, in accordance with the treatment of Section 4.

An analogous phenomenon (existence of domain walls with zero energy and, therefore, of different ground states with regular structures) takes place also in the model (1.1) with a triangular lattice and $f = \frac{1}{4}$ (Figs. 5d–5f). Here the shaded triangles correspond to charges $+\frac{3}{4}$ and light – to $-\frac{1}{4}$. The bonds for which the absolute value of $(\Delta\varphi)_{\mathbf{rr}'}$ is equal to $\pi/2$ are shown by bold lines and by thin—those for which $(\Delta\varphi)_{\mathbf{rr}'} = 0$.

For the model (1.1) with a square lattice and $f = \frac{1}{4}$ there are two different types of a regular ground state (Figs. 5g and 5h). In contrast to the previously considered cases, neither of them can be obtained as a regular set of the domain walls on the background of another ground state.

In these and many other anomalous cases the type of ordering at low temperatures also can be found only when spin waves' free energy is taken into account.

## 7. A HONEYCOMB LATTICE: A CASE OF SMALL *f*

Let us express $f$ as an irreducible fraction: $p/q$. A question of the types and temperatures of phase transitions in the case of $q \gg 1$ seems to be of a certain interest. It was discussed by Teitel and Jayaprakash[3] and by Shih and Stroud,[4] the conclusions these authors have arrived at being different. Teitel and Jayaprakash[3] are of the opinion that for large $q$ the transition temperature decreases as $1/q$, but Shih and Stroud[4] do not see reasons for such a drastic dependence. As in both cases the temperatures of transitions were obtained not by analyzing the topological defects' energies but in a more indirect manner, we feel it appropriate to resume this question.

Consider the model (1.1) with a honeycomb lattice and $f = 1/(3n^2)$. The corresponding Coulomb gas is defined on a triangular lattice. In the ground state the charges $1 - f$ form a regular triangular lattice $\sqrt{3}\,n \times \sqrt{3}\,n$, the other sites being occupied by charges $-f$ (the case of

$n = 1$ was considered in Section 5). We shall obtain rough estimates of different phase transition temperatures, not taking into account the interaction of topological defects belonging to different classes.

Due to independence of excessive charges interaction of the type of the regular structure forming the ground state (see Section 2) the dissociation temperature of integer excessive charges can be taken to be $f$ independent and can be estimated by a value obtained by Kosterlitz and Thouless[18] for an ordinary (without frustration) $XY$ model:

$$T_{\mathrm{BKT}} \approx \frac{\pi}{2} J \tag{7.1}$$

If one takes into account only neutral domain walls and those with minimal charge density, the classification of the domain walls turns out to be the same as in the case $n = 1$ (Section 5). They can be treated as the ones formed by links of a length $\sqrt{3}\, n$, each link having a charge $\pm 1/(3n)$ (Fig. 6). If neighboring links join at the angle of 120° their charges are opposite in sign, in contrast to cases of 60° and 180° joints. Thus the minimal fractional charge localized at the defect of a domain wall regular structure (Fig. 6) equals $\pm 1/(3n)$. The dissociation temperature for pairs of such charges can be estimated as

$$T_{\mathrm{BKT}}^{\mathrm{frac}} \approx \frac{\pi}{2(3n)^2} J = \frac{\pi}{6} fJ \tag{7.2}$$

We will estimate the temperature of the phase transition associated with restoring of the sublattices' equivalence by calculating the energy of

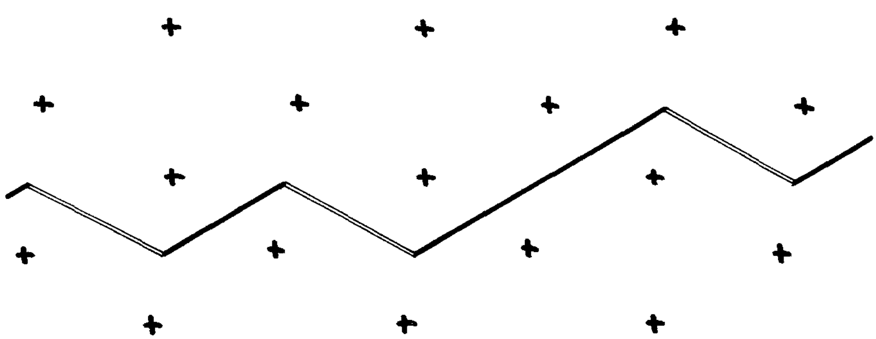

Fig. 6. Domain wall with kink in the Coulomb gas corresponding to the model (1.1) with a honeycomb lattice and $f = 1/(3n^2)$. Only positive charges $1 - f$ are shown. The charge $+1/(3n)$ is localized on the kink.

the simplest point defect, which consists of the charge $1-f$ shifted to the neighboring site:

$$T_c \sim \frac{2\pi^2}{3} fJ \tag{7.3}$$

Comparing (7.1) to (7.3) one can conclude that in the case $f \ll 1$ $(n \gg 1)$ the dissociation of pairs of integer excessive charges (ordinary BKT transition) cannot occur as an independent phase transition, because it is preceded by domain wall transition. The same dependence on $f$ entering (7.2) and (7.3) does not allow us to unambigously find out whether the dissociation of pairs of fractional vortices will take place at higher temperatures than the restoring of sublattices' equivalence or simultaneously with it. The comparison of the numerical factors entering (7.2) and (7.3) favors the latter opportunity. The temperature of the phase transition (there is only one in this case) turns out to be proportional to $f$, as has been suggested by Teitel and Jayaprakash.[3] It seems that the mean field approximation of Shih and Stroud[4] is appropriate only for describing the ordinary BKT transition, which cannot occur for small $f$. In such a case, the reasons for the other kind of dependence of the transition temperature on $f$ obtained by these authors[4] become clear.

Note that (7.2) and (7.3) are valid for $f = 1/(3n^2)$ only. In other cases, when, for instance, the ground state is a regular structure $q \times q$ (see Ref. 3) the temperatures of the transitions considered may prove to be quite different.

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## NOTE ADDED IN PROOF

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