Quantum-mechanical tunneling with dissipation in a sloping sinusoidal potential

S. E. Korshunov

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR, Moscow
(Submitted 28 October 1986)
Zh. Eksp. Teor. Fiz. **92**, 1828–1833 (May 1987)

A quantum-mechanical system described by an effective Caldeira-Leggett action in which dissipation is taken into account by means of a term that is nonlocal in imaginary time is considered. In the limit of large viscosity the probability of tunneling between minima of a sinusoidal potential with an arbitrary slope is calculated in the exponential approximation. The dependences found in various limiting cases agree with previously known dependences.

1. INTRODUCTION

Although more than twenty years have passed since Feynman and Vernon[1] suggested that in the study of the effect of dissipation on the properties of quantum systems one can imitate the influence of the medium by interaction with a heat bath (an infinite set of oscillators), the application of this method to the study of the effect of dissipation on quantum-mechanical tunneling began only comparatively recently. It was first used for a calculation of the probability of tunneling in a dissipative system by Caldeira and Leggett,[2] who studied the decay of a metastable state at temperature $T = 0$. Subsequently, this type of approach was extended to finite temperatures[3–5] and also to the case of a degenerate or quasidegenerate potential.[6–14] In all these papers a system describable by an effective Euclidean action

$$A[q(t)]=\int\limits_{-1/2T}^{1/2T}\int dt\,dt'\,G_0^{-1}(t-t')\,q(t)\,q(t')+\int\limits_{-1/2T}^{1/2T} dt\,V[q(t)] \tag{1a}$$

was considered, where the propagator G_0 in the Fourier representation has the form

$$G_0^{-1}=m\omega^2+\eta|\omega| \tag{1b}$$

(a quantum system with "nonlocal" dissipation). Here and below, m is the effective mass, η is the effective viscosity, and the Planck constant $\hbar$, like the Boltzmann constant k_B, is set equal to unity.

A systematic calculation of the tunneling probability (with allowance for the pre-exponential factor) has been carried out previously only for a potential $V(q)$ in the form of a cubic parabola,[4] corresponding to the decay of a metastable state into the continuum. For a degenerate potential only the case of low viscosity ($\eta^2 \ll m|V''|$), when the influence of the viscosity on the form of the extremal trajectory and on the pre-exponential factor can be neglected, has been considered.[6–9] In the present paper the tunneling probability will be calculated for a sinusoidal potential with arbitrary slope (see Fig. 1):

$$V(q)=V\sin\left(\frac{2\pi}{q_0}q\right)-\frac{2\pi}{q_0}Fq,\qquad |F|<V \tag{1c}$$

in the limit of large viscosity

$$\eta^2\gg mU,\qquad U=(2\pi/q_0)^2(V^2-F^2)^{1/2}. \tag{2}$$

To calculate the tunneling probability we shall make use of the well known formula of the exponential approximation (see, e.g., Ref. 4):

$$\Gamma=Be^{-A}, \tag{3}$$

where

$$A=A[Q]-A[q_m], \tag{4}$$

$$B=\left\{\frac{\left[\int\limits_{-1/2T}^{1/2T}\frac{dt}{2\pi}\left(\frac{dQ}{dt}\right)^2\right]\det\left(\frac{\delta^2A}{\delta q^2}\right)_{q=q_m}}{\left|\det'\left(\frac{\delta^2A}{\delta q^2}\right)_{q=Q(t)}\right|}\right\}^{1/2}, \tag{5}$$

in which $Q(t)$ is the cyclically closed extremal trajectory (bounce trajectory) constituting the saddle point in the functional space of trajectories, and q_m is the value of q at the minimum (corresponding to the initial state of the system) of the potential (1c). In the calculation of one of the determinants the zero eigenvalue should be omitted, and this is indicated by a prime.

We emphasize that in the case of a degenerate potential ($F = 0$) at temperature $T\to 0$ the tunneling will certainly have an incoherent character (purely exponential relaxation) only if the dimensionless viscosity $\alpha = (q_0^2/2\pi)\eta$ exceeds unity.[8] Only in this case can we introduce a probability (rather than an amplitude) of tunneling, and for $\alpha \gg 1$ this probability can be calculated using the formulas (3)–(5). As the temperature rises or the difference in the depths of

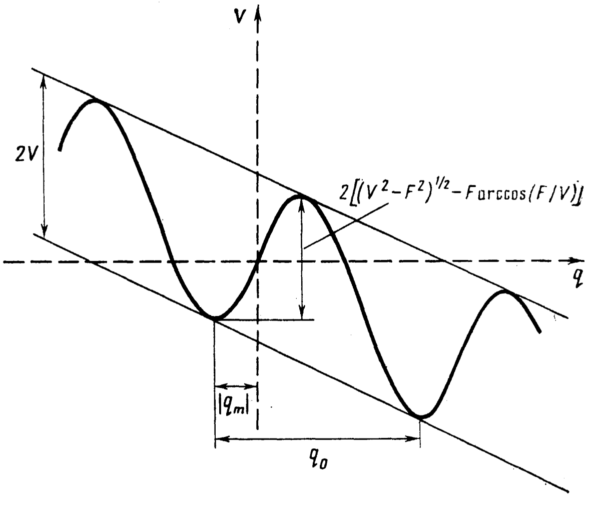

FIG. 1.

0038-5646/87/051025-04$04.00

neighboring minima becomes larger the region of applicability of this description is extended to lower values of the viscosity.[13,14]

Among real physical systems possessing an effective action close in structure to (1) we can mention a Josephson junction shunted by a normal resistance.[15,16] In this case a nonzero value of F corresponds to a finite current flowing across the junction. This application is of special interest, since recently there has been active experimental investigation of macroscopic quantum tunneling in Josephson junctions.[17] Another example is the motion of a particle interacting with a medium in a crystalline potential under the action of an external force. It is assumed, in particular in Ref. 18, that such an approach makes it possible to describe the diffusion of muons in metals.

2. THE EXTREMAL TRAJECTORY

Variation of the action (1) leads to the following equation for the classical (extremal) trajectories:

$$-m\frac{d^2}{dt^2}q+\eta T\int_{-1/2T}^{1/2T}dt'\,\mathrm{ctg}[\pi T(t-t')]\frac{dq}{dt'}=\frac{2\pi}{q_0}\left[-V\cos\left(\frac{2\pi}{q_0}q\right)+F\right]. \quad (6)$$

The integral in the left-hand side of (6) should be understood in the sense of the principal value. The tunneling probability is determined by the cyclically closed trajectory on which q passes twice through the maximum of the potential (1c). When the condition (2) is fulfilled the first term in the left-hand side of (6) can be omitted, after which the solution of (6) of interest to us can be found exactly:

$$q=\frac{q_0}{2\pi}\left[-2\,\mathrm{arctg}\frac{b+\cos(2\pi Tt)}{a}+\arcsin\frac{F}{V}+\frac{\pi}{2}\right]\equiv Q(t). \quad (7)$$

Here

$$a=\mathrm{sh}\,\theta\cos\varphi,\quad b=\mathrm{ch}\,\theta\sin\varphi,$$

$$\mathrm{cth}^2\,\theta=(V^2-F^2)/(\alpha T)^2,\quad \mathrm{tg}\,\varphi=F/\alpha T.$$

On the trajectory (7) the action takes the value

$$A[Q]=\alpha\ln\frac{V^2}{F^2+(\alpha T)^2}+2\alpha-\frac{(V^2-F^2)^{1/2}}{T}+\frac{F}{T}\left(2\varphi-\frac{\pi}{2}-\arcsin\frac{F}{V}\right) \quad (8)$$

(where again we have omitted a small term proportional to m), while on the trajectory $q=q_m=-(q_0/2\pi)\arccos(F/V)$ the action takes the value

$$A[q_m]=[-(V^2-F^2)^{1/2}+F\arccos(F/V)]/T. \quad (9)$$

Subtracting (9) from (8), we find the value of the exponent in (3):

$$A=\alpha\ln\frac{V^2}{F^2+(\alpha T)^2}+2\alpha-\frac{2F}{T}\mathrm{arctg}\frac{\alpha T}{F}. \quad (10)$$

Here we also calculate the value of the normalization factor appearing in (5):

$$\int_{-1/2T}^{1/2T}\frac{dt}{2\pi}\left(\frac{dQ}{dt}\right)^2=2(V^2-F^2)^{1/2}(V^2-F^2-\alpha^2T^2)/\eta V^2. \quad (11)$$

The quantity A was found for tunneling from left to right, to a lower-lying minimum (see the figure, in which $F>0$). In the calculation of the tunneling probability in the opposite direction to that in (10) we must add $2\pi|F|/T$, corresponding to another branch of the arctangent.

An extremal trajectory of the form (7) exists only for $T\leqslant T_0=(V^2-F^2)^{1/2}/\alpha$. For $T>T_0$, and also for $T=T_0$, the tunneling probability is determined by the trajectory $q=q_m$, corresponding to the regime of thermal (activation) tunneling. In this case

$$A=\frac{2}{T}\left[(V^2-F^2)^{1/2}-F\arccos\frac{F}{V}\right]. \quad (12)$$

The quantity A, whose temperature dependence in the ranges $T\leqslant T_0$ and $T\geqslant T_0$ is given by the formulas (10) and (12), respectively, has a break at the point $T=T_0$. However, as shown in Ref. 4, when one takes the fluctuations into account systematically by going beyond the limits of the Gaussian approximation, the singularity in $\Gamma(T)$ is smoothed out. Below we shall not be interested in the immediate vicinity of the point $T=T_0$.

3. THE PRE-EXPONENTIAL FACTOR

The calculation of the determinants appearing in (5) requires the second variation of the action and diagonalization of the resulting linearized operator. For the trajectory (7) the corresponding equation for the eigenfunctions has the form

$$-m\frac{d^2}{dt^2}\tilde q+\eta T\int_{-1/2T}^{1/2T}dt'\,\mathrm{ctg}[\pi T(t-t')]\frac{d}{dt'}\tilde q+2\pi T\eta\left\{\mathrm{ctg}\,\theta-\frac{2\,\mathrm{sh}\,\theta[\mathrm{ch}\,\theta+\sin\varphi\cos(2\pi Tt)]}{a^2+[b+\cos(2\pi Tt)]^2}\right\}\tilde q=\Lambda\tilde q. \quad (13)$$

Here we have kept the term proportional to m, which it will be necessary to take into account only for the large eigenvalues. For $m=0$ Eq. (13) has the following complete set of eigenfunctions and eigenvalues:

$$\tilde q_{0,1}=\frac{\cos\chi_{0,1}-\sin\chi_{0,1}\cos(2\pi Tt)}{a^2+[b+\cos(2\pi Tt)]^2}, \quad (14)$$

$$\Lambda_{0,1}=U\left\{\frac{1}{2}-\frac{F^2+(\alpha T)^2}{V^2}\mp\left[\frac{1}{4}+\frac{F^2(F^2+(\alpha T)^2)(V^2-F^2-(\alpha T)^2)}{V^4(V^2-F^2)}\right]^{1/2}\right\}, \quad (15)$$

$$\tilde q_{-1}=\frac{\sin(2\pi Tt)}{a^2+[b+\cos(2\pi Tt)]^2},\quad \Lambda_{-1}=0, \quad (16)$$

$$\tilde q_n=\cos\left[2\pi Tt(n-2)-2\,\mathrm{arctg}\frac{\sin\varphi+\mathrm{ch}\,\theta\cos(2\pi Tt)}{\mathrm{sh}\,\theta\sin(2\pi Tt)}\right] \quad (17)$$

$$\tilde q_{-n}=\sin\left[2\pi Tt(n-2)-2\,\mathrm{arctg}\frac{\sin\varphi+\mathrm{ch}\,\theta\cos(2\pi Tt)}{\mathrm{sh}\,\theta\sin(2\pi Tt)}\right], \quad (18)$$

$$\Lambda_n=\Lambda_{-n}=U+2\pi T\eta(n-2),\quad n\geqslant 2, \quad (19)$$

where

$$\chi_{0,1}=\mathrm{arctg}\left\{\frac{\mathrm{ch}\,\theta}{\sin\varphi\,\mathrm{ch}\,2\theta}\times\left[\frac{\Lambda_{0,1}}{U}(\mathrm{ch}^2\,\theta-\sin^2\varphi)-(\mathrm{sh}^2\,\theta+\sin^2\varphi)\right]\right\},$$

$$U=(2\pi/q_0)^2(V^2-F^2)^{1/2}.$$

The eigenfunctions (17) and (18) are the wavefunctions of the semiclassical approximation, which, in the case under consideration ($m = 0$), gives the exact answer. The numeration of the eigenfunctions has been chosen in such a way that the absolute value of the label is equal to the number of pairs of zeros; negative labels correspond to odd eigenfunctions, and non-negative labels to even eigenfunctions. As we should expect, among the eigenvalues there is one negative (Λ_0) and one zero (Λ_{-1}) eigenvalue. The eigenvalues $\Lambda_{\pm n}$ with $n \geqslant 2$ were found to be the same as in the problem of a cubic parabola.[4]

For large labels of the eigenvalues one must include in the analysis the first term in the left-hand side of (13). In this case the last term in the left-hand side can be taken into account in the framework of perturbation theory,[4] and this gives

$$\Lambda_{\pm n} = U + 2\pi T(n-2)\eta + (2\pi T n)^2 m. \tag{20}$$

Since for small n the expression (20) goes over into (19), following Larkin and Ovchinnikov[4] we shall use formula (20) for all values $n \geqslant 2$.

For the trajectory $q = q_m$ the eigenfunction of the operator $\delta^2 A/\delta q^2$ are ordinary sines and cosines, and the eigenvalues are

$$\Lambda_{\pm n}' = U + 2\pi T n\eta + (2\pi T n)^2 m. \tag{21}$$

Substituting (11), (15), (20), and (21) into (5), we obtain

$$B = \left\{\frac{2q_0^2(V^2-F^2)}{(2\pi)^2\eta[F^2+(\alpha T)^2]}\right\}^{1/2}(2\pi T)^2 m \frac{\Gamma(2-n_1')\Gamma(2-n_2')}{\Gamma(1-n_1)\Gamma(1-n_2)},$$

where $n_{1,2}$ and $n_{1,2}'$ are the solutions of the quadratic equations $\Lambda_n = 0$ and $\Lambda_n' = 0$, respectively. Expanding the gamma function in the limit of small m, we have

$$B = \left(\frac{q_0}{2\pi}\right)^3\left[\frac{2\eta^7}{F^2+(\alpha T)^2}\right]^{1/2}\frac{1}{m^2}. \tag{22}$$

4. DISCUSSION

Thus, we have shown that, for a dissipative quantum system with the potential (1c), when the criterion (2) is fulfilled the exponential and pre-exponential factors in the expression (3) for the tunneling probability are given by the formulas (10) and (22), respectively. Of course, here the mass m should be not so small that the semiclassical approximation ceases to be applicable.

The problem considered allows us to pass to the limit of a potential of the cubic-parabola type. If we let q_0 go to infinity, having set

$$V+F = 24V_1(q_0/2\pi q_1)^3, \quad V-F = {}^3/_2 V_1(q_0/2\pi q_1),$$

the potential (1c) goes over into

$$\tilde{V}(q) = V_1[{}^3/_2(q/q_1) - 2(q/q_1)^3]. \tag{23}$$

The potential (23) is a cubic parabola in which the minimum and maximum are at a distance q_1 from each other and the barrier height is egual to V_1. In this limit process the expressions (10) and (22) go over into

$$A = \frac{\pi\eta q_1^2}{2} - \frac{1}{2}\left(\frac{\pi\eta q_1^2}{3}\right)^3\left(\frac{T}{V_1}\right)^2, \quad B = \frac{(2\eta^7)^{1/2} q_1^3}{12V_1 m^2},$$

which coincide with the results obtained directly for a cubic parabola by Larkin and Ovchinnikov.[3,4]

In the case of a degenerate potential ($F = 0$) we shall have

$$\Gamma(T) = \left(\frac{q_0}{2\pi}\right)^3 \frac{(2\eta^7)^{1/2}}{Vm^2} e^{-2\alpha}\left(\frac{\alpha T}{V}\right)^{2\alpha-1}. \tag{24}$$

The power exponent $2\alpha - 1$ in the temperature dependence $\Gamma(T)$ coincides with that found earlier in the opposite limiting case $mU \gg \eta^2$ (Ref. 8), and this indicates that this exponent is universal. We emphasize that the expression (24) is valid in the entire region $0 \leqslant T \leqslant T_0$, and not only for $T \to 0$. The presence of the factor $\exp(-2\alpha)$ ensures for $\alpha \gg 1$ that Γ is also small when $T \sim T_0$. We note that, contrary to the assertion in Ref. 11, the expression (24) does not contain Δ_0^2 (where Δ_0 is the tunneling amplitude in the absence of dissipation) as a universal factor. This, incidentally, is fully understandable, since the extremal trajectory (7) differs strongly from the extremal trajectory in the absence of dissipation.

In the incoherent-tunneling regime under consideration the diffusion coefficient can be expressed in a trivial manner in terms of the probability Γ of tunneling to the neighboring minimum: $D = q_0^2\Gamma$ (Ref. 9), so that $D \propto T^{2\alpha-1}$.

At $T = 0$ the $\Gamma(F)$ dependence is determined by the same power exponent as the temperature dependence of the expression (24):

$$\Gamma(F) = \left(\frac{q_0}{2\pi}\right)^3 \frac{(2\eta^7)^{1/2}}{Vm^2}\left(\frac{F}{V}\right)^{2\alpha-1}, \tag{25}$$

just as in the case of a low viscosity.[9–11] For $F \ll V$ we can introduce a crossover temperature $T_* = F/\alpha$ such that for $T \ll T_*$ the expression (25) applies, while for $T_* \ll T < T_0$ the dependence (24) holds.

Whatever the relative magnitudes of the parameters, the values of both A and B decrease monotonically with increase of the temperature; here Γ increases everywhere except in the small region

$$\alpha < {}^3/_2, \quad T < x(\alpha)F/\alpha,$$

where $x(\alpha)$ is the positive solution of the equation

$$x - \frac{1}{2\alpha}\frac{x^3}{1+x^2} = \operatorname{arctg} x.$$

Here, however, it is not entirely clear whether the related formulas (3)–(5) are applicable for at least a qualitative description of the behavior of the tunneling probability so close to the critical point $\alpha = 1$, $T = 0$.

[1] R. P. Feynman and F. L. Vernon, Jr., Ann. Phys. (N.Y.) **24**, 118 (1963).
[2] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1982); Ann. Phys. (N.Y.) **149**, 374 (1983); **153**, 445 (1984).
[3] A. I. Larkin and Yu. N. Ovchinnikov, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 322 (1983) [JETP Lett. **37**, 382 (1983)].
[4] A. I. Larkin and Yu. n. Ovchinnikov, Zh. Eksp. Teor. Fiz. **86**, 719 (1984) [Sov. Phys. JETP **59**, 420 (1984)].
[5] H. Grabert, U. Weiss, and P. Hanggi, Phys. Rev. Lett. **52**, 2193 (1984); H. Grabert and U. Weiss, Phys. Rev. Lett. **53**, 1787 (1984); Z. Phys. B **56**, 171 (1984); H. Grabert, P. Olschowski, and U. Weiss, Phys. Rev. B **32**, 3348 (1985).
[6] S. Chakravarty, Phys. Rev. Lett. **49**, 681 (1982); A. J. Bray and M. A. Moore, Phys. Rev. Lett. **49**, 1545 (1982).
[7] A. Schmid, Phys. Rev. Lett. **51**, 1506 (1983); S. A. Bulgadaev, Pis'ma Zh. Eksp. Teor. Fiz. **39**, 264 (1984) [JETP Lett. **39**, 315 (1984)]; Zh.

Eksp. Teor. Fiz. **90**, 634 (1986) [Sov. Phys. JETP **63**, 369 (1986)].
[8]S. Chakravarty and A. J. Leggett, Phys. Rev. Lett. **52**, 5 (1984).
[9]H. Grabert and U. Weiss, Phys. Lett. **108A**, 63 (1985).
[10]H. Grabert and U. Weiss, Phys. Rev. Lett. **54**, 1605 (1985).
[11]M. P. A. Fisher and A. T. Dorsey, Phys. Rev. Lett. **54**, 1609 (1985).
[12]U. Weiss, P. Riseborough, P. Hanggi, and H. Grabert, Phys. Lett. **104A**, 10, 492 (1984).
[13]A. Garg, Phys. Rev. B **32**, 4746 (1985).
[14]A. J. Leggett, S. Chakravarty, A. T. Dorsey, *et al.*, Preprint No. P/85/10/158, University of Illinois at Urbana-Champaign (1985).
[15]V. Ambegaokar, U. Eckern, and G. Schön, Phys. Rev. Lett. **48**, 1745 (1982); U. Eckern, G. Schön, and V. Ambegaokar, Phys. Rev. B **30**, 6419 (1984).
[16]A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **85**, 1510 (1983) [Sov. Phys. JETP **58**, 876 (1983)]; Phys. Rev. B **28**, 6281 (1983).
[17]R. F. Voss and R. A. Webb, Phys. Rev. Lett. **47**, 265 (1981); L. D. Jackel; J. P. Gordon, E. L. Hu, R. E. Howard, L. A. Fetter ,D. M. Tennant, R. W. Epworth, and J. Kurkijarvi, Phys. Rev. Lett. **47**, 697 (1981); S. Washburn, R. A. Webb, R. F. Voss, and S. M. Faris, Phys. Rev. Lett. **54**, 2712 (1985); M. H. Devoret, J. M. Martinis, and J. Clarke, Phys. Rev. Lett. **55**, 1908 (1985).
[18]M. P. A. Fisher and W. Zwerger, Phys. Rev. B **32**, 6190 (1985).

Translated by P. J. Shepherd