Multiparticle exchange processes and the phase transition of an electron crystal in a magnetic field

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An exactly soluble model is proposed which is isomorphic with the partition function of non-intersecting rings on the lattice, which describe multiparticle as well as two-particle exchange processes in an electron crystal in a magnetic field. A similar partition function was introduced by Kivelson *et al.* [Phys. Rev. Lett. **56**, 873 (1986); Phys. Rev. B**36**, 1620 (1987)] and by Bychkov [JETP Lett. **43**, 388 (1986)] to explain the fractional quantum Hall effect.

In the work of Kivelson *et al.*[1] an interpretation of the fractional quantumHall effect was proposed as a manifestation of multiparticle ring exchange processes of a phase transition into a two-dimensional electron crystal. The phase-transition point corresponds to the appearance of rings of infinite length which lead to the possibility of charge transfer without the formation of excitations. The authors[1] showed that an energy shift produced by ring exchange processes in the ground state of the electron crystal can be described by a functional integral reducible to the partition function of non-intersecting rings on the lattice, each of which enters with the weight

$$W = Be^{-\alpha L}(-1)^{L+1}(e^{2\pi i\Phi}+e^{-2\pi i\Phi}), \qquad (1)$$

where L is the ring length (in terms of the lattice constant) and Φ is the magnitude of the magnetic flux through it (in units of the flux quantum). Clearly, for a regular two-dimensional lattice $\Phi = fS$, where S is the number of sites encompassed by the ring.

The factor $e^{-\alpha L}$ would also arise in the case of a Bose crystal and is an exponential whose argument is the real part of the Euclidian action on the extremal trajectory. The factor $(-1)^{L+1}$ is the parity of the ring exchange of L electrons. The terms $\exp(\pm 2\pi i\Phi)$ arise from the multiplication of the factors $\exp(ie(\hbar c)^{-1}\int \mathbf{A}d\mathbf{l})$ for all the particles participating in the ring exchange (whence $f = 1/2\nu$, where ν is the filling factor of the lowest Landau level). The different signs correspond to different directions of motion. The pre-exponential factor B must be evaluated as the ratio of fluctuation determinants.

The supposition was put forward[1] that B is negative in the problem of the fractional quantum Hall effect, allowing the authors, for $B = -e^{-(\Delta\alpha)L}$, to go from rings with the weight (1) to a discrete Gaussian model with imaginary field and, further, to a frustrated XY model. No actual calculations of the fluctuation determinants that confirm these suggestions were carried out, however. Statistical weights of the form (1) with $B = 1$ arise in Bychkov's paper,[2] where a similar partition function is obtained by different means. Bearing in mind the lack of proof of the assumption that B is negative and also the possible applicability of a similar description to other fermion crystals, an exactly soluble model is proposed here, isomorphic with the partition function of rings with statistical weights (1) and with $B = 1$ (fluctuation corrections to α are assumed included in the definition of α).

We determine the partition function Z in terms of an integral over the anticommuting fields ψ and $\bar{\psi}$ specified at sites **i** of an arbitrary lattice:

$$Z = \prod_{\mathbf{i}}\left(\iint d\psi_{\mathbf{i}}\, d\bar{\psi}_{\mathbf{i}}\right)\prod_{\mathbf{jk}}[(1+e^{-\alpha+i\varphi_{\mathbf{jk}}}\bar{\psi}_{\mathbf{j}}\psi_{\mathbf{k}})(1+e^{-\alpha+i\varphi_{\mathbf{kj}}}\psi_{\mathbf{k}}\psi_{\mathbf{j}})]$$

$$\times \exp\sum_{\mathbf{l}}\bar{\psi}_{\mathbf{l}}\psi_{\mathbf{l}}, \qquad (2)$$

where

$$\varphi_{\mathbf{jk}} = e(\hbar c)^{-1}\int_{\mathbf{k}}^{\mathbf{j}} \mathbf{A}\, d\mathbf{l}, \qquad \varphi_{\mathbf{kj}} \equiv -\varphi_{\mathbf{jk}},$$

with the product taken over all pairs of sites coupled by bonds. Expanding the infinite product contained in Eq. (2) into a sum and carrying out the integration, we obtain for Z a representation whose different terms make up the partition function of non-intersecting rings with weight (1), where $B = 1$ and $\Phi = \Sigma\varphi$. It is not difficult to be ccnvinced of this by noting that the factor $e^{-\alpha+i\varphi}$ corresponds to each bond through which a ring passes, while to each empty site corresponds the factor 1 ensured by the last factor in Eq. (2). The factor $(-1)^{L+1}$ is produced by the exchange properties of the fields ψ and $\bar{\psi}$. The non-zero statistical weights $-e^{-2\alpha}$ are also ascribed to pseudorings (rings with $L = 2$, $S = 0$), whic are represented graphically by segments of unit length. Such pseudorings can be compared with pair exchange processes. A certain deficiency of the model is that the corresponding statistical weight cannot be varied as an independent parameter but, since it turned out to be independent of f, its absolute value will not influence the nature of the depences of different quantities on the magnetic field.

In this way the partition function (2) is the partition function of non-intersecting and non-self-intersecting rings (and pseudorings) with weight (1) and with $B = 1$. If desired, different values of α for different bonds could be introduced into Eq. (2). For any (not necessarily two-dimensional or regular) lattice this partition function can in principle be evaluated exactly, for which purpose it is sufficient to introduce into Eq. (2) factors preceding the argument of the exponential, and this leads to

$$Z = \det(1+e^{-\alpha}\hat{L}), \qquad (3)$$

where the operator $\hat{L}$ has non-zero matrix elements

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$L_{\mathbf{jk}} = \exp i\varphi_{\mathbf{jk}}$ only for those pairs of sites **j** and **k** which are coupled by bonds. It follows from the form of Eq. (3) that the phase-transition point corresponds to $\alpha = \alpha_c = \ln(-\lambda_{\min})$, where $\lambda_{\min}$ is the minimum eigenvalue of the operator $\hat{L}$. An analogous phase transition curve, determined by the band edge of the operator $\hat{L}$, arises in an analysis within the framework of mean field theory of the problem of the superconducting transition of a mesh of thin wires placed in a magnetic field.

For a regular triangular lattice (which is especially interesting in the context of the problem of the fractional quantum Hall effect) the band structure formed by the eigenvalues of the operator $\hat{L}$ for different values of the magnetic field, was studied by Claro and Wannier[4] when investigating the problem of the splitting of the Landau levels of a single electron in a periodic potential. It follows from their results that although the periodic function $\alpha_c(f)$ is also continuous, it has a fractal structure and undergoes breaks (discontinuities in the derivative) for rational values of f. The most clearly marked peaks are for half-integer f ($\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, etc.) at which $\alpha_c(f)$ reaches its maximum $\alpha_c = (n + \frac{1}{2}) = \ln 6 \approx 1.792$. The next most significant peaks for $f = (2n+1)/4$ ($\nu = 2/(2n+1)$, $\alpha_c = \frac{1}{2}\ln 12 \approx 1.242$) and $f = n$, $n \pm \frac{1}{6}$ ($\nu = 1/2n$, $3/(6n \pm 1)$, $\alpha_c = \ln 3 \approx 1.099$) are already much more weakly expressed ($\min \alpha_c(f) \approx 0.96$).

For $\alpha > \alpha_c(f)$ the dependence of $\ln Z \equiv \mathrm{Sp}\ln(1 + e^{-\alpha}\hat{L})$ on α and f can be found by expanding in powers of $\hat{L}$. It follows from Eq. (3) that for $\alpha \to \alpha_c + 0$ the singularity of Z will be of the same type as is the fluctuation correction to the partition function in the Landau mean-field theory. It is interesting that for all values of f except integer and half-integer, a similar singularity in behavior will also occur at a number of other points, $\alpha_{cm} < \alpha$, associated with the landing of the quantity $-e^{-\alpha}$ on the boundary of one band or another in the spectrum of the operator $\hat{L}$. The elucidation of the physical meaning of these features is, so far, an unsolved problem.

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