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Instability of Superfluid Helium Free Surface in the Presence of Heat Flow.

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Abstract. – The spectrum of surface waves in superfluid helium is calculated for the case when there is a constant nondissipative heat flow in helium parallel to the surface. The presence of such a flow decreases the frequency of surface waves and can even make the flat surface unstable.

Experimental investigations of standing second-sound waves in superfluid helium with free surface have shown that increase in wave amplitude leads to instability of the surface [1]. Oscillations whose frequency is much less than that of the second-sound wave develop. It seems worth mentioning that in the experiment the second sound is emitted by an a.c.-heated electric wire. Therefore not only a second sound but also a steady heat flow is emitted. One becomes naturally interested in how the constant nondissipative heat flow parallel to the surface influences the spectrum of surface waves on superfluid and whether it can make the flat surface unstable. The present letter is devoted to investigation of this problem.

In the general case a superfluid liquid with free surface can be described by the Lagrangian

$$L = \int \mathrm{d}x \int \mathrm{d}y \left[\int_{-\infty}^{\zeta} (E + \rho\dot{\alpha} + S\dot{\beta} + \rho g z) + \sigma\sqrt{1 + (\nabla_\parallel \zeta)^2} \right], \tag{1}$$

where ρ stands for the mass density, $S \equiv \rho s$ is the enthropy density,

$$E = \frac{\rho}{2}(\nabla\alpha)^2 + (\nabla\alpha) S\nabla\beta + E_0(\rho, S, S\nabla\beta) \tag{2}$$

is the energy density of the superfluid; α and β are velocity potentials describing superfluid and relative motion:

$$\boldsymbol{v}_\mathrm{s} = \nabla\alpha\,, \qquad \boldsymbol{v}_\mathrm{n} - \boldsymbol{v}_\mathrm{s} = \frac{S}{\rho_\mathrm{n}}\nabla\beta\,, \tag{3}$$

ζ is the deviation of the surface from the plane $z = 0$ and σ is the surface energy. Equation (1) is a generalization of the Lagrangian introduced by Pokrovsky and Khalatnikov [2] for the superfluid without free surface. Variation of eq. (1) with respect to different variables gives not only the correct equations of motion in the bulk but also the correct boundary conditions.

When considering long-wavelength surface oscillations with frequencies much less than that of the sound and the second-sound waves with the same wave vector one can neglect the fluctuations of density and enthropy. In this case E_0 reduces to the pure kinetic part of the internal energy:

$$E_0 = \frac{1}{2\rho_n}(S\nabla\beta)^2 \equiv \frac{\rho_n}{2}(\boldsymbol{v}_n - \boldsymbol{v}_s)^2 \tag{4}$$

and E to the total kinetic energy

$$E = \frac{\rho}{2}(\nabla\alpha + s\nabla\beta)^2 + \frac{\rho_s\rho}{2\rho_n}(s\nabla\beta)^2 \equiv \frac{\rho_s}{2}v_s^2 + \frac{\rho_n}{2}v_n^2 . \tag{5}$$

Here $\rho_s \equiv \rho - \rho_n$.

Variation of eq. (1) with respect to α and β in the bulk gives then the equations of motion in the form

$$\Delta\alpha = 0 ; \qquad \Delta\beta = 0 . \tag{6}$$

For the investigation of the uniform heat flow influence on surface waves one should choose the solutions of eqs. (6) in the form

$$\alpha = (\boldsymbol{v}_s^0 \boldsymbol{r}) + \tilde{\alpha} , \tag{7}$$

$$\beta = -\frac{1}{s}(\boldsymbol{v}_s^0 \boldsymbol{r}) + \tilde{\beta} , \tag{8}$$

$$\tilde{\alpha}, \tilde{\beta} \propto \exp[i\boldsymbol{q}\boldsymbol{r}_\parallel + qz - i\omega t] . \tag{9}$$

The first terms on the r.h.s. of eqs. (7) and (8) are describing the background motion of the liquid with superfluid velocity $\boldsymbol{v}_s^0$ and normal velocity $\boldsymbol{v}_n^0 = -(\rho_n/\rho_s)\boldsymbol{v}_s^0$, that is with no mass flow and constant nondissipative heat flow parallel to the surface $((v_s^0)_z = (v_n^0)_z = 0)$. The second terms correspond to the surface wave with wave vector $\boldsymbol{q}$ along the surface and frequency ω.

Variation of Lagrangian (1) with respect to surface variables gives three boundary conditions describing i) mass conservation, ii) heat conservation and iii) relation between pressure and curvature (Laplace equation). For our simple limit of $\rho = \text{const}$ and $s = \text{const}$ in linear approximation they can be written in the form

$$\frac{\partial}{\partial z}(\tilde{\alpha} + s\tilde{\beta}) = \dot{\zeta} , \tag{10}$$

$$\frac{\partial}{\partial z}\tilde{\alpha} + \frac{S}{\rho_n}\frac{\partial}{\partial z}\tilde{\beta} - (\boldsymbol{v}_n^0 \nabla_\parallel \zeta) = \dot{\zeta} , \tag{11}$$

$$\frac{\partial}{\partial t}(\tilde{\alpha} + s\tilde{\beta}) - \frac{\rho_s}{\rho_n}(\boldsymbol{v}_n^0 \nabla_\parallel \zeta)\tilde{\beta} + \left(g + \frac{\sigma}{\rho}k^2\right)\zeta = 0 . \tag{12}$$

After substitution of eq. (9) into eqs. (10)-(12) a system of the algebraic equations is obtained. The condition of their mutual compatibility gives the dispersion relation of surface waves:

$$\omega^2 = gq + \frac{\sigma}{\rho} q^3 - \frac{\rho_n}{\rho_s} (\boldsymbol{v}_n^0 \boldsymbol{q})^2 . \tag{13}$$

Thus with increase of v_n^0 the frequency of surface waves is decreased and for large enough heat flow the surface becomes unstable. This happens at

$$v_n^0 = v_{nc}^0 \equiv \left(2 \frac{\rho_s}{\rho_n}\right)^{1/2} \left(\frac{g\sigma}{\rho}\right)^{1/4} . \tag{14}$$

For example for $T = 1.5\,\mathrm{K}$, $\rho_s/\rho_n \sim 10$ and $v_{nc}^0 \sim 15\,\mathrm{cm/s}$.

To conclude, we have calculated the frequency of surface waves in superfluid liquid in the presence of uniform heat flow parallel to the surface. For large enough flow the flat surface becomes unstable. The discovered instability is analogous to the instability of tangential discontinuity in ordinary liquids [3]. The results of our calculations cannot be compared directly with the experiment [1] where the heat flow is combined with the second-sound wave but they definitively show that for surface instability the presence of uniform heat flow is an important factor.

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