Stability analysis of a two-dimensional uniaxial vortex glass

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The simplest model of a vortex glass is considered which is applicable for the description of a two-dimensional uniaxial vortex crystal formed by the fluxon lines in a large area Josephson junction with inhomogeneous width. The analysis is performed in replica representation in terms of a free-energy functional which depends on the renormalized correlation function. The properties of different solutions of the Dyson equation are considered, the main attention being devoted to investigating the stability of these solutions. In particular the solution with the one-step replica symmetry breaking which corresponds to the absolute maximum of free energy is shown to be always stable (when it exists at all). The unimportance of higher-order corrections for the form of the asymptotic behavior of the correlation function in the phase with the one-step replica symmetry breaking is also demonstrated.

I. INTRODUCTION

The discovery of the high-T_c superconconducting materials have essentially increased the possibilities for the experimental observations of various phenomena related with the presence of vortices in superconductors (vortex lattice melting, pinning, creep and so on). This has led to the active development of theoretical investigation of these phenomena and the appearance of many new ideas (for a recent review see Ref. 1). In particular a suggestion has been made that at low temperatures a phase should exist in which the motion of the vortices is quenched by disorder and therefore the linear resistance is absent.[2,3] The properties of this phase (including the multitude of metastable states separated by the diverging barriers) are expected to be more or less analogous to those of widely discussed infinite-range spin-glass models[4] and therefore it is usually called a vortex glass.

It has been suggested[2,5] that the simplest model which allows one to analyze the large scale properties of a vortex crystal (or charge-density wave) interacting with a random potential can be described by the Hamiltonian

$$H=\int d^D\mathbf{r}\left[\frac{J}{2}(\nabla u)^2+V_1(\mathbf{r})\cos u(\mathbf{r})+V_2(\mathbf{r})\sin u(\mathbf{r})\right], \tag{1}$$

where the variable u represents the smoothly varying component of the displacement.

The first term in Eq. (1) describes the elastic energy of a vortex crystal which is chosen in the simplest possible form, whereas the second term describes the most relevant contribution to the interaction of a vortex crystal with a random potential.[2,5] The distribution of a random potential has to be invariant with respect to arbitrary shift of u, therefore the distribution of the functions $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$ can depend only on $V_1^2(\mathbf{r})+V_2^2(\mathbf{r})$. The simplest choice is to consider the Gaussian distribution with local correlations the parameters of which are defined by

$$\overline{V_i(\mathbf{r})}=0;\quad \overline{V_i(\mathbf{r})V_j(\mathbf{r}')}=2Y\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\quad (i,j=1,2). \tag{2}$$

Here and further on an overbar designates the average over disorder, whereas the average over thermal fluctuations will be denoted by the angular brackets.

The model (1) takes into account only the uniaxial displacements of the vortices. When $D=2$ it can be used to describe a vortex crystal formed by the fluxon lines in a large area Josephson junction to which the magnetic field is applied in parallel to the junction plane, the random potential being related with the inhomogenities in the width of the junction. The case $D=3$ corresponds to the description of a vortex crystal in a superconductor with well developed layered structure in situation when the magnetic field is parallel to the layers. The uniaxial model (1) can be generalized to incorporate the multicomponent displacements[6,7] but the properties associated with the glassy behavior can be expected to be present already in the simplest uniaxial case.

In the general situation the most important drawback of the model (1) is that it does not take into account the possibility of formation of dislocations. But in the case of a two-dimensional uniaxial vortex crystal formed by the fluxon lines the dislocations would correspond to the end points of these lines and therefore cannot exist. In the present article we concentrate exclusively on the two-dimensional uniaxial case for which the description of flux lattice pinning with the help of Hamiltonian (1) is rather accurate.

Another possible application of the two-dimensional version of model (1) is the description of crystal growth in presence of quenched disorder.[8] But probably the most important reason for the investigation of this model is that it is one of the few low-dimensional systems which presumably demonstrate the glassy properties but allow for application of different kinds of analytical treatment which take into account the fluctuations.

In terms on a unit vector

$$\mathbf{d}=(\cos u,\sin u), \tag{3}$$

the Hamiltonian (1) reduces to the form

$$H=\int d^D\mathbf{r}\left[\frac{J}{2}(\nabla\mathbf{d})^2+\mathbf{V}(\mathbf{r})\mathbf{d}(\mathbf{r})\right], \tag{4}$$

corresponding to the XY model with random field $\mathbf{V}=(V_1,V_2)$. However since we assume the variable u to be continuous and uniquely defined the model (1) should be identified with the random field XY model in which the creation of vortices is prohibited. In terms of a vortex crystal the vortices of the XY model correspond to dislocations.

The first wave of interest to the two-dimensional version of model (1) has developed precisely in the context of the random-field XY model.[9–14] After Houghton *et al.*[9] had demonstrated that the anharmonic terms in Eq. (1) are irrelevant at high temperatures, but become relevant at low temperatures, the more systematic renormalization-group description of the phase transition has been developed by Cardy and Ostlund[10] and Goldschmidt and Houghton.[11] The difference between the two phases manifests itself in the asymptotic behavior of the correlation function

$$C(\mathbf{R})=\overline{\langle[u(\mathbf{r}+\mathbf{R})-u(\mathbf{r})]^2\rangle}, \tag{5}$$

which in the high-temperature phase diverges logarithmically (like in the absence of disorder) whereas in the low-temperature phase the renormalization-group equations predict that $C(R)$ has to diverge as the square of logarithm.[8,12]

The predictions of the renormalization-group approach developed in the framework of replica representation turned out to be in agreement with the results of the real-space renormalization procedure suggested by Villain and Fernandes[13] and with the dynamic renormalization-group analysis of Goldschmidt and Schaub.[14]

Recently it has been shown with the help of self-consistent harmonic approximation (SCHA) that in the low-temperature phase the replica symmetry breaking can occur.[6,7] Such a possibility is not taken into account in the renormalization-group calculations[9–11] since they explicitly assume the situation to be replica-symmetric. SCHA predicts that the asymptotic behavior of the correlation function (5) should be logarithmic also in the low-temperature phase, the only difference with the high-temperature phase being in the temperature dependence of the prelogarithmic factor.[6,7] The analogous behavior of the static correlation function in the low-temperature phase is also predicted by the self-consistent dynamic analysis,[15] which in contrast to the dynamic renormalization group[14,16] allows (and requires) for the possibility of the fluctuation-dissipation theorem breaking.

So far there is no complete agreement on the meaning of the contradiction between the predictions of the renormalization-group analysis and SCHA. The tendency exists to assume[17,18] that since the renormalization scheme is a much more systematic approach than the variational procedure incorporated in SCHA, the former should be more trusted than the latter.

In the present work we suggest (Sec. II) that both approaches can be understood in the framework of the same general scheme which consists in considering a free-energy as a functional of the renormalized correlation function.[19,20] This correlation function has to satisfy a Dyson equation which can be obtained by a variation of a free energy functional and contains the infinite sequence of diagrams. The discrepancy between the predictions of different approaches appears because in the renormalization-group calculations of Refs. 9–11 and in the SCHA of Refs. 6 and 7 simply the different solutions of this complex equation are considered which can coexist at the same temperature. And although one of these solutions is known more accurately than the other this does not prove that the second one does not exist at all.

To choose which of the two solutions really describes the properties of the system additional arguments may be needed, the simplest of which may be related with the stability of these solutions. The solution which correctly describes the properties of the system cannot be unstable with respect to small variations.[4]

Giamarchi and Le Doussal[7] have shown that in the framework of SCHA the replica-symmetric solution is unstable and therefore the choice between the two solutions should be made in favor of the replica symmetry breaking solution. But since the form of the replica-symmetric solution is known much more accurately that the simplest possible form suggested by SCHA, it is certainly important to check if this solution remains unstable when the renormalization effects are taken into account. We discuss this problem in Sec. IV after rederiving in Sec. III the renormalization-group equations in the form which simplifies their application in the investigation of the stability problem.

In Sec. IV we consider the two sources for the instability of the replicon modes (only one of which can be discussed in terms of SCHA) and show that although each of them in the absence of the renormalization corrections leads to the instability of the replica-symmetric solution, the inclusion of the renormalization effects removes the instability. In the situation when both sources of the instability are present simultaneously (as they always are in the real system) our approach allows only to conclude that the replica-symmetric solution has to remain stable at least in some finite interval of temperatures below the transition temperature.

The stability of the other solution, which is the solution with the one-step replica symmetry breaking, is investigated in Sec. V C where we find the interval of the possible sizes of the block in which this solution is stable. The extremal point corresponding to the absolute maximum of free energy always belongs to this interval. Thus, at least in some part of phase diagram both coexisting solutions of the Dyson equation are stable so the choice between them can rely only on the comparison of their free energies.

The results presented in Sec. V (devoted to the replica symmetry breaking solution) include also the more detailed description of the phase diagram predicted by SCHA which shows that the phase-transition line is split by a singular point into two segments with different critical behavior. And in Sec. V D we demonstrate that the higher-order corrections do not change the form of the asymptotic behavior of the correlation function corresponding to the solution with the one-step replica symmetry breaking with respect to what is predicted by SCHA. Section VI is devoted to a short discussion of the results.

II. FREE ENERGY AS A FUNCTIONAL OF THE RENORMALIZED CORRELATION FUNCTION

The replica approach[21,22] is based on a simple identity:

$$\ln Z=\lim_{n\to 0}\frac{1}{n}(Z^n-1), \tag{6}$$

which allows one to calculate the average over disorder of the free energy of the system by calculating the average of the partition function of n replicas of the same system (at the end of calculation n should be set equal to zero). For the model (1) the result of such averaging can be rewritten as a partition function corresponding to the Hamiltonian:

$$H=\int d^2\mathbf{R}\left[\frac{J}{2}\sum_a (\nabla u^a)^2 - Y\sum_{a\neq b}\cos(u^a-u^b)\right], \quad (7)$$

where the replica indices a and b run from 1 to n. In the second sum the constant term corresponding to $a=b$ has been for simplicity omitted. We assume that the factor $1/T$ is included into the definition of the original Hamiltonian (1), so $J\propto 1/T$ and $Y\propto 1/T^2$.

Hamiltonian (7) can be rewritten in a more general form as

$$H=\frac{1}{2}\int\frac{d^2\mathbf{q}}{(2\pi)^2}\sum_{a,b}[G_0^{-1}(\mathbf{q})]^{ab}u^a(\mathbf{q})u^b(-\mathbf{q}) - Y\int d^2\mathbf{R}\sum_{\mathbf{s}}\exp\left[i\sum_a s^a u^a(\mathbf{R})\right], \quad (8)$$

where

$$[G_0^{-1}(\mathbf{q})]^{ab}=G_0^{-1}(\mathbf{q})\delta^{ab};\quad G_0^{-1}(\mathbf{q})=Jq^2 \quad (9)$$

and the set of n-dimensional vectors $\mathbf{s}$ consists of $n(n-1)$ vectors:

$$s^a=\begin{cases}1 & \text{for } a=\alpha,\\ -1 & \text{for } a=\beta,\\ 0 & \text{for } a\neq\alpha,\beta,\end{cases} \quad (10)$$

which can be numbered by two indices α and β $(1\leqslant\alpha,\beta\leqslant n)$ not equal to each other.

Expansion of the partition function corresponding to the Hamiltonian (8) in powers of the anharmonic term allows to get rid of the continuous variables $u^a(\mathbf{R})$ (by performing a Gaussian integration) and to rewrite it as a partition function of a Coulomb gas:

$$H_{\mathrm{CG}}=\frac{1}{2}\sum_{i,j}\sum_{a,b}\mathbf{s}_i^a G_0^{ab}(\mathbf{R}_i-\mathbf{R}_j)\mathbf{s}_j^b+N\ln(1/Y) \quad (11)$$

formed by the vector charges $\mathbf{s}_i$.[10] Here N is the total number of charges in a given configuration.

On the other hand the analogous expansion (in powers of the fugacity Y) of the free energy produces a series with the same structure as has been constructed by Amit *et al.* for the free energy of the one-component sine-Gordon model.[23] This expansion

$$\widetilde{F}\{\hat{G}_0(\mathbf{q}),Y\}=\sum_{p=0}^{\infty}\widetilde{F}_p\{\hat{G}_0(\mathbf{q}),Y\};\quad \widetilde{F}_p\{\hat{G}_0(\mathbf{q}),Y\}\propto Y^p, \quad (12)$$

in which the zeroth-order term

$$\widetilde{F}_0\{\hat{G}_0(\mathbf{q})\}=\frac{1}{2}\int\frac{d^2\mathbf{q}}{(2\pi)^2}\ln\det\hat{G}_0^{-1}(\mathbf{q}) \quad (13)$$

is the free energy corresponding to the harmonic part of the Hamiltonian (8), can be used as a formal definition of a free energy as a functional of the bare correlation function $G_0^{ab}(\mathbf{q})$ and fugacity Y. Instead of introducing diagrammatical notation it will be more convenient in the following to work directly with explicit expressions for the low-order terms in different expansions.

It is not hard to notice that since the first term in Eq. (8) is harmonic the renormalized correlation function of the replicated system

$$G^{ab}(\mathbf{q})\equiv\langle u^a(\mathbf{q})u^b(-\mathbf{q})\rangle \quad (14)$$

can be found by calculating a variation of the free energy with respect to the inverse of the bare correlation function:

$$G^{ab}(\mathbf{q})=8\pi^2\frac{\delta\widetilde{F}}{\delta[G_0^{-1}(\mathbf{q})]^{ab}}. \quad (15)$$

The form of Eq. (15) implies that in terms of the thermodynamics $[G_0^{-1}(\mathbf{q})]^{ab}$ and $G^{ab}(\mathbf{q})$ are conjugate to each other and therefore with the help of the Legendre transformation the free energy can be expressed as a functional of $G^{ab}(\mathbf{q})$:[19]

$$F=F_0\{\hat{G}_0(\mathbf{q}),\hat{G}(\mathbf{q})\}+F_{\mathrm{int}}\{\hat{G}(\mathbf{q}),Y\};$$

$$F_{\mathrm{int}}=\sum_{p=1}^{\infty}F_p\{\hat{G}(\mathbf{q}),Y\};\quad F_p\propto Y^p \quad (16)$$

In terms of the diagrammatic expansion the Legendre transformation of Ref. 19 reduces to the exclusion of the diagrams which can be decomposed into two parts by breaking two lines, whereas the form of the zeroth-order term is changed into

$$F_0=\frac{1}{2}\int\frac{d^2\mathbf{q}}{(2\pi)^2}(\ln\det\hat{G}^{-1}(\mathbf{q}) - \mathrm{Sp}\{[\hat{G}^{-1}(\mathbf{q})-\hat{G}_0^{-1}(\mathbf{q})]\hat{G}(\mathbf{q})\}). \quad (17)$$

Only the zeroth-order term in expansion (16) depends on the bare correlation function $G_0^{ab}(\mathbf{q})$, whereas all the higher-order terms, starting from

$$F_1=-Y\sum_{\mathbf{s}_1}\exp\left[-\frac{1}{2}\sum_{a,b}s_1^a G^{ab}(\mathbf{R}=0)s_1^b\right] \equiv -Y\sum_{\mathbf{s}_1}\exp\left(-\frac{1}{2}\mathcal{G}_{11}\right) \quad (18)$$

and

$$F_2=-\frac{Y^2}{2}\sum_{\mathbf{s}_1,\mathbf{s}_2}\int d^2\mathbf{R}_2\exp\left(-\frac{1}{2}\mathcal{G}_{11}-\frac{1}{2}\mathcal{G}_{22}\right) \times\left[\exp(-\mathcal{G}_{12})-\frac{1}{2}\mathcal{G}_{12}^2-1\right] \quad (19)$$

depend only on the renormalized correlation function $G^{ab}(\mathbf{q})$, or to put it more precisely on the expressions of the form

$$\mathscr{G}_{ij}=\sum_{a,b} s_i^a G^{ab}(\mathbf{R}_i-\mathbf{R}_j)s_j^b, \tag{20}$$

which were introduced in Eqs. (18) and (19) to make them more compact.

In terms of the Coulomb gas Eq. (20) describes the interaction energy of two vector charges $\mathbf{s}_i$ and $\mathbf{s}_j$, whereas the energy E_0 of a single charge is given by

$$E_0=\frac{1}{2}\mathscr{G}_{ii}=\frac{1}{2}\sum_{a,b} s_i^a G^{ab}(\mathbf{R}=0)s_i^b. \tag{21}$$

Since we keep explicitly the summation over $\mathbf{s}_i$ in different expressions it is always possible to identify these expressions with a particular combination of charges to which they correspond. For example Eq. (18) describes the contribution to the free energy from the unbound charges whereas Eq. (19) includes both the contribution from the bound pairs of charges and from the interactions between the unbound charges.

The main property of the functional (16) is that at its stationary points, that is when $G^{ab}(\mathbf{q})$ satisfies the Dyson equation

$$[G^{-1}(\mathbf{q})]^{ab}=[G_0^{-1}(\mathbf{q})]^{ab}+\Sigma^{ab}\{\hat{G}(\mathbf{q})\}, \tag{22}$$

which can be obtained by the variation of Eq. (16) with respect to $G^{ab}(\mathbf{q})$, the self-energy part $\Sigma^{ab}\{\hat{G}(\mathbf{q})\}$ being related to a variation of $F_{\rm int}$:

$$\Sigma^{ab}\{\hat{G}(\mathbf{q})\}=8\pi^2\frac{\delta F_{\rm int}}{\delta G^{ab}(\mathbf{q})}; \tag{23}$$

the free energy defined by Eq. (16) coincides with the free energy defined by the original functional (12).[19] In the general case Eq. (22) can have different solutions corresponding to different values of free energy. That means that the summation of some sequence of divergent diagrams in the functional (12) cannot be performed in the unique way and the result can depend on the regularization. In that case the use of the functional depending on the renormalized correlation function simplifies the consideration removing some unphysical divergencies right from the beginning.

For the case of the Hamiltonian (8) the first two terms in the expansion for the self-energy part $\Sigma^{ab}(\mathbf{q})$ have the form

$$\Sigma_1^{ab}(\mathbf{q})=8\pi^2\frac{\delta F_1}{\delta G^{ab}(\mathbf{q})}=Y\sum_{\mathbf{s}_1} s_1^a s_1^b \exp\left(-\frac{1}{2}\mathscr{G}_{11}\right) \tag{24}$$

and

$$\Sigma_2^{ab}=8\pi^2\frac{\delta F_2}{\delta G^{ab}(\mathbf{q})}=Y^2\sum_{\mathbf{s}_1,\mathbf{s}_2}\int d^2\mathbf{R}_2\exp\left(-\frac{1}{2}\mathscr{G}_{11}-\frac{1}{2}\mathscr{G}_{22}\right)\times\left\{s_1^a s_1^b\left[\exp(-\mathscr{G}_{12})-\frac{1}{2}\mathscr{G}_{12}^2-1\right]+s_1^a s_2^b\cos(\mathbf{qR})[\exp(-\mathscr{G}_{12})-\mathscr{G}_{12}]\right\}. \tag{25}$$

Since G^{ab} enters $F_{\rm int}$ only through the combinations of the form (20) and all the vectors $\mathbf{s}$ obey the relation $\Sigma_a s^a=0$, the expression for the self-energy part $\Sigma^{ab}(\mathbf{q})$ always satisfies the relation

$$\sum_a \Sigma^{ab}(\mathbf{q})=\sum_b \Sigma^{ab}(\mathbf{q})=0, \tag{26}$$

which holds also in any particular order in fugacity.

In Refs. 6 and 7 the same problem has been approached by calculating a variational free energy

$$F_{\rm VAR}=F_{\rm TR}+\langle H-H_{\rm TR}\rangle_{\rm TR}, \tag{27}$$

corresponding to the Hamiltonian (7) with the help of the harmonic trial Hamiltonian:

$$H_{\rm TR}=\frac{1}{2}\int\frac{d^2\mathbf{q}}{(2\pi)^2}\sum_{a,b}[G^{-1}(\mathbf{q})]^{ab}u^a(\mathbf{q})u^b(-\mathbf{q}). \tag{28}$$

In Eq. (27) $F_{\rm TR}$ stands for the free energy for the trial Hamiltonian (28) and $\langle\ldots\rangle_{\rm TR}$ for the thermodynamic average calculated with the help of $H_{\rm TR}$. Both terms can be calculated exactly. Such an approach, which is also known as the self-consistent harmonic approximation (SCHA), has proved to give a correct qualitative description of both phases of the two-dimensional sine-Gordon model.[24] Recently it has been applied by Mezard and Parisi[25] to the problem of fluctuating manifold in random media.

Substitution of Eq. (28) into Eq. (27) shows that the expression for the variational free energy coincides with the sum of the two lowest-order terms in the expansion (16):

$$F_{\rm VAR}=F_0+F_1, \tag{29}$$

and therefore the application of this particular form of the variational approach (to the system which allows to express its free energy as a functional of the renormalized correlation function) should be considered not as an uncontrollable approximation based on unjustified assumptions but rather as the first step of a more general and systematic treatment. The importance of this step is related to the fact that the self-consistent equation

$$[G^{-1}(\mathbf{q})]^{ab}=[G_0^{-1}(\mathbf{q})]^{ab}+\Sigma_1^{ab}\{\hat{G}(\mathbf{q})\} \tag{30}$$

obtained by the variation of the expression (29) for the free energy [which obviously is just a simplest possible truncation of the general Dyson equation (22)] can have not only replica-symmetric but also replica symmetry breaking solutions.[6,7]

We discuss the properties of the replica symmetry breaking solutions of Eq. (30) in Sec. V after investigating the stability of the replica-symmetric solution. But it has been necessary to explain the meaning of SCHA before that, since so far the stability problem has been discussed only in the framework of this particular approximation.[7]

III. THE REPLICA-SYMMETRIC SOLUTION

According to Eq. (26) when the symmetry with respect to a permutation of replicas (the replica symmetry) is not broken the only possible form of the self-energy matrix is

$$\Sigma^{ab}(\mathbf{q})=(n\delta^{ab}-1)\sigma(\mathbf{q}). \tag{31}$$

For such a form of $\Sigma^{ab}(\mathbf{q})$ the inversion of Eq. (22) in the limit of $n\to 0$ gives

$$G^{ab}(\mathbf{q})=G_0(\mathbf{q})\delta^{ab}+G_0^2(\mathbf{q})\sigma(\mathbf{q}). \tag{32}$$

Since the sum of the components s_i^a of any vector charge $\mathbf{s}_i$ is equal to zero, the second term in the right-hand side (rhs) of Eq. (32) drops out from any expression of the form (20), which therefore is reduced to

$$\mathcal{G}_{ij}=(\mathbf{s}_i\mathbf{s}_j)G_0(\mathbf{R}_i-\mathbf{R}_j). \tag{33}$$

The form of Eq. (33) shows that the interaction of vector charges in the replica-symmetric case always remains unrenormalized. The origin of this property can be traced back to the statistical invariance of the initial problem with respect to the arbitrary uniform translation in u.

As a consequence $\sigma(\mathbf{q})$ drops out from all the terms of the expansion for the free energy (with the exception of the zeroth-order term) and from the rhs of the Dyson equation (22), which in the replica-symmetric case can be reduced to the scalar form

$$\sigma(\mathbf{q})=-8\pi^2\frac{\delta f_{\rm int}}{\delta G_0(\mathbf{q})}, \tag{34}$$

where

$$f_{\rm int}=\lim_{n\to 0}\frac{1}{n}F_{\rm int} \tag{35}$$

is a disorder-induced contribution to the free energy per replica. Therefore the problem of solving the Dyson equation in the replica-symmetric case does not exist—in that case this equation is reduced to the explicit expression for the self-energy function. But since this expression contains an infinite number of terms there remains a problem of the summation of all the essential contributions to it.

In the replica-symmetric case the expression for the energy of a single vector charge $\mathbf{s}$

$$E_0=G_0(\mathbf{R}=0)=\int\frac{d^2\mathbf{q}}{(2\pi)^2}G_0(\mathbf{q}), \tag{36}$$

is logarithmically divergent and therefore the first-order contribution to the replica-symmetric self-energy function

$$\sigma_1(\mathbf{q})=2Y\exp[-E_0] \tag{37}$$

in the limit of infinite system size is equal to zero. The first nonvanishing contribution to $\sigma(\mathbf{q})$ appears in the second-order in fugacity:

$$\sigma_2(\mathbf{q})=Y^2\int d^2\mathbf{R}(2-2\cos\mathbf{qR})W(R), \tag{38}$$

where $W(R)=\exp[-E_p(R)]$ is the statistical weight which can be associated with the neutral pair of charges whose total energy $E_p(R)$ is given by the same expression

$$E_p(R)=2[G_0(0)-G_0(R)] \tag{39}$$

as the correlation function $C(R)$ of the pure system.

For $R\gg a$ (where a is the cutoff length defined by the form of the cutoff in momentum space) $E_p(R)$ behaves logarithmically

$$E_p(R)\approx 4K\ln\frac{R}{a};\quad K=\frac{1}{4\pi J}\propto T, \tag{40}$$

and therefore $W(R)$ is characterized by an algebraic behavior:

$$W(R)\approx\left(\frac{a}{R}\right)^{4K}. \tag{41}$$

For example, if a sharp cutoff at $|\mathbf{q}|=q_c$ is assumed a and q_c are related as

$$aq_c=2e^{-\gamma_e}\approx 1.123, \tag{42}$$

where γ_e is the Euler's constant.

The only terms which survive in the $n\to 0$ limit and give a nonvanishing contribution to $\sigma(\mathbf{q})$ correspond to the neutral pairs of charges ($\mathbf{s}_1+\mathbf{s}_2=0$). This is also true in all the higher orders: when $G_0(\mathbf{R}=0)$ is divergent only the terms corresponding to the neutral combinations of charges are finite.

For small q the expression (38) can be approximated as

$$\sigma_2(q)\approx Bq^2, \tag{43}$$

where

$$B=\pi Y^2\int dR\,R^3W(R). \tag{44}$$

Comparison with Eq. (41) shows that the integral in Eq. (44) is convergent only for $K>1$. For $K\leq 1$ this integral diverges and therefore the approximation (43) is no longer valid. Actually for $\frac{1}{2}<K<1$ the straightforward calculation of the integral in Eq. (38) gives

$$\sigma_2(q)\approx\eta q^{4K-2} \tag{45}$$

but since the same divergence appears also in the higher orders of the expansion, this answer has to be corrected.

Let us call the pair of charges $\mathbf{s}_j$ and $\mathbf{s}_l$, the total charge of which $\mathbf{s}_i=\mathbf{s}_j+\mathbf{s}_l$ belongs to the original set of elementary charges (10), the reducible pair. Since for the distances much larger than $\mathbf{R}_j-\mathbf{R}_l$ the reducible pair is indistinguishable from the single charge $\mathbf{s}_i$, it is possible to take into account the important sequence of higher-order diagrams by introducing the scale dependent fugacity $Y(R)$:

$$Y(R)=Y+2\pi(n-2)\int_0^R dR'\,Y^2(R')W^{1/2}(R'). \quad (46)$$

The factor $(n-2)$ in the rhs of Eq. (46) stands for the number of reducible pairs which can imitate given elementary charge $\mathbf{s}_i$, whereas the weight factor $W^{1/2}(R)$ appears because the energy of the reducible pair of charges differs from the energy of a single charge by the amount equal to $\frac{1}{2}E_p(\mathbf{R}_j-\mathbf{R}_l)$.

By differentiating both sides of Eq. (46) it can be reduced to the differential equation

$$\frac{dY(R)}{dR}=-4\pi R Y^2(R)W^{1/2}(R), \quad (47)$$

(where we have put n to zero) the solution of which can be written as

$$Y(R)=[Y^{-1}+4\pi I(R)]^{-1};\quad I(R)=\int_0^R dR'R'W^{1/2}(R'). \quad (48)$$

For $W(R)$ of the form (41) the function $I(R)$ can be approximated as

$$I(R)\approx\int_a^R dR'R'\left(\frac{a}{R'}\right)^{2K} = \begin{cases} \dfrac{a^2}{2(1-K)}\left[\left(\dfrac{R}{a}\right)^{2-2K}-1\right] & \text{for } K\neq 1 \\ a^2\ln(R/a) & \text{for } K=1. \end{cases} \quad (49)$$

The choice of the lower integration limit in Eq. (49) implies that the bare value of fugacity Y can be associated with the smallest possible length in the system—the cutoff length a.

The more familiar form of the differential equation describing the renormalization of the fugacity[10]

$$\frac{dy}{dl}=(2-2K)y+2\pi(n-2)y^2 \quad (50)$$

[where $l=\exp(R/a)$] can be recovered by introduction of a rescaled fugacity:

$$y=a^2(R/a)^{2-2K}Y(R), \quad (51)$$

but in the following it will more convenient to work with the unrescaled variables since all the expressions which are of interest to us have a more transparent form in terms of these variables.

If the results of the field-theoretical analysis of Goldschmidt and Houghton[11] are translated into the language of the unrescaled variables the main conclusion is that they reveal no other divergencies in addition to those which can be described by the renormalization of fugacity according to Eq. (47). Therefore to take into account all the important higher-order corrections to any expression it is sufficient to substitute the constant fugacity by the renormalized one. For example Eq. (38) for the second-order contribution to the self-energy part should be substituted by

$$\sigma(\mathbf{q})=\int d^2\mathbf{R}(2-2\cos\mathbf{qR})Y^2(R)W(R). \quad (52)$$

It can be seen from Eq. (49) that the behavior of $Y(R)$ for $R\to\infty$ depends essentially on whether K is smaller or larger than 1. For $K>1$ the renormalized fugacity $Y(R)$ tends to a finite limit as R goes to infinity. That means that Eq. (52) corresponds to the same asymptotic behavior of $\sigma(\mathbf{q})$ as suggested by Eq. (43) with B given by

$$B=\pi\int_0^\infty dR\;R^3Y^2(R)W(R). \quad (53)$$

On the other hand for $K<1$ the renormalized fugacity $Y(R)$ tends to zero as $(a/R)^{2-2K}$ which makes the integral in Eq. (53) diverge logarithmically. In the nontruncated expression (52) this divergence is cut off at $R\sim q^{-1}$ giving in the $q\to 0$ limit

$$B\approx\frac{(1-K)^2}{4\pi}\ln\frac{1}{aq}. \quad (54)$$

For $K\to 1+0$ the value of B tends to the finite limit $Ya^2/4$.

To avoid confusion maybe it is worthwhile to emphasize once again that we are using the renormalized but unrescaled fugacity $Y(R)$, whereas the behavior of the rescaled (and renormalized) fugacity $y(l)$ is exactly the opposite: $y(l)$ goes to zero (for $l\to\infty$) in the high-temperature phase whereas in the low-temperature phase it has a finite limit.[10,11]

In the disordered systems it is important to distinguish between the full correlation function (5) which in terms of the replicated system is given by

$$C(\mathbf{R})=\int\frac{d^2\mathbf{q}}{(2\pi)^2}(2-2\cos\mathbf{qR})\lim_{n\to 0}\left[\frac{1}{n}\sum_a G^{aa}(\mathbf{q})\right] \quad (55)$$

and its irreducible part

$$C_{\rm ir}(\mathbf{R})=\overline{\langle[u(\mathbf{r}+\mathbf{R})-u(\mathbf{r})]^2\rangle}-\overline{\langle u(\mathbf{r}+\mathbf{R})-u(\mathbf{r})\rangle^2} = \int\frac{d^2\mathbf{q}}{(2\pi)^2}(2-2\cos\mathbf{qR})\lim_{n\to 0}\left[\frac{1}{n}\sum_{a,b} G^{ab}(\mathbf{q})\right]. \quad (56)$$

Substitution of Eq. (32) into Eq. (56) shows that in the replica-symmetric case the irreducible part of the correlation function remains exactly the same as in the absence of disorder:

$$C_{\rm ir}(R)=\int\frac{d^2\mathbf{q}}{(2\pi)^2}(2-2\cos\mathbf{qR})G_0(\mathbf{q})\approx 4K\ln(R/a) \quad (57)$$

that is unrenormalized. On the other hand, the long-distance behavior of the reducible part of the correlation function:

$$C_r(R)=\overline{\langle u(\mathbf{r}+\mathbf{R})-u(\mathbf{r})\rangle^2} = \int\frac{d^2\mathbf{q}}{(2\pi)^2}(2-2\cos\mathbf{qR})\lim_{n\to 0}\left[\frac{1}{n}\sum_{a\neq b} G^{ab}(\mathbf{q})\right] \quad (58)$$

is determined by $\sigma(\mathbf{q})$:

$$C_r(R)=\int\frac{d^2\mathbf{q}}{(2\pi)^2}(2-2\cos\mathbf{qR})G_0^2(\mathbf{q})\sigma(\mathbf{q}),\tag{59}$$

and therefore is qualitatively different for $K>1$ and $K<1$. In the high-temperature phase ($K>1$) in which for $q\to 0$ the approximation (43) can be used the reducible part of the correlation function diverges logarithmically:

$$C_r(R)\approx 4K_1\ln(R/a);\quad K_1=\frac{B}{4\pi J^2},\tag{60}$$

whereas in the low-temperature phase B is itself diverging according to Eq. (54) and the asymptotic form of C_r acquires additional logarithmical factor

$$C_r(R)\approx 2K^2(1-K)^2\ln^2(R/a).\tag{61}$$

Note that for $K\to 1$ the prefactor in Eq. (61) coincides with the one which can be deduced from the renormalization equations of Goldschmidt and Houghton[11] which have been derived in a much more systematic way than presented here and accurately take into account the explicit form of the cutoff, both in the coordinate and in the momentum space. On the other hand, the value of the prefactor cited in Refs. 17 and 26 as being universal is larger by a factor of 4.

In the terms of the vector Coloumb gas the phase transition between the two phases described above is very peculiar since in both of them the charges are bound in neutral pairs and their interaction is exactly the same. Usually the phase transition in a Coulomb gas can be associated with the dissociation of the neutral pairs of charges which leads to formation of a "plasma" phase in contrast to a "dielectric" one in which all the charges are bound in pairs. In the present model this can happen only if the replica symmetry breaking is allowed.

IV. STABILITY ANALYSIS OF THE REPLICA-SYMMETRIC SOLUTION

To investigate the stability of any solution of the Dyson equation (22) one has to consider the second variation of the free-energy functional (16). The result can be again expressed as an expansion in powers of fugacity:

$$L^{ab,cd}(\mathbf{q},\mathbf{q}')\equiv 2(2\pi)^4\frac{\partial^2 F}{\partial G^{ab}(\mathbf{q})\partial G^{cd}(\mathbf{q}')}=\sum_{p=0}^{\infty}L_p^{ab,cd}(\mathbf{q},\mathbf{q}');\quad L_p^{ab,cd}(\mathbf{q},\mathbf{q}')\propto Y^p.\tag{62}$$

In order to simplify some equations we have included additional factor $2(2\pi)^4$ in the definition of the Hessian $L^{ab,cd}(\mathbf{q},\mathbf{q}')$.

The zeroth-order term in Eq. (62) is diagonal in momentum:

$$L_0^{ab,cd}(\mathbf{q},\mathbf{q}')=(2\pi)^2\delta(\mathbf{q}-\mathbf{q}')[G^{-1}(\mathbf{q})]^{da}[G^{-1}(\mathbf{q})]^{bc},\tag{63}$$

whereas the first-order term does not depend on momenta at all:

$$L_1^{ab,cd}(\mathbf{q},\mathbf{q}')=-\frac{Y}{2}\sum_{\mathbf{s}} s^a s^b s^c s^d\times\exp\left[-\frac{1}{2}\sum_{e,f}s^e G^{ef}(\mathbf{R}=0)s^f\right].\tag{64}$$

In the replica-symmetric case Eq. (64) reduces to

$$L_1^{ab,cd}(\mathbf{q},\mathbf{q}')=-\frac{1}{2}P^{abcd}\sigma_1,\tag{65}$$

where the matrix

$$P^{abcd}=\frac{1}{2}\sum_{\mathbf{s}} s^a s^b s^c s^d\tag{66}$$

is symmetric with respect to all possible permutations of indices, whereas σ_1 is the first-order contribution (37) to the self-energy function $\sigma(\mathbf{q})$.

The stability of replica-symmetric solution has been considered by Giamarchi and Le Doussal[7] in the framework of SCHA. These authors have noticed that although in two dimensions E_0 is given by the logarithmically divergent expression and therefore in the limit of infinite system size σ_1 is always equal to zero, the presence of σ_1 in Eq. (65) can still be of importance if some regularization procedure is used. It will be convenient to rederive here the results of Ref. 7 in the form which allows for the inclusion of the renormalization effects.

To check the stability of any solution of the Dyson equation one has to look for the lowest eigenvalue of the equation:

$$\lambda g^{ab}(\mathbf{q})=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\sum_{c,d}L^{ab,cd}(\mathbf{q},\mathbf{q}')g^{cd}(\mathbf{q}').\tag{67}$$

In SCHA only the two lowest-order contributions to $L^{ab,cd}(\mathbf{q},\mathbf{q}')$ [given by Eqs. (63) and (65)] should be taken into account. The eigenvalues of the matrix $\hat{P}$ defined by Eq. (66) are equal to 0, n, $2n$, and 2. The last one has degeneracy $n(n-3)/2$ and corresponds to the family of eigenstates which includes all the matrices Ψ^{ab} satisfying the constraints:

$$\Psi^{ab}=\Psi^{ba};\quad \Psi^{a=b}=0;\quad \sum_a\Psi^{ab}=\sum_b\Psi^{ab}=0.\tag{68}$$

These eigenstates are usually referred to as the replicon modes. In the limit of $n\to 0$ they are the only modes which can be dangerous for the stability of a replica-symmetric solution with respect to a replica symmetry breaking.

For replicon modes, that is for $g^{ab}(\mathbf{q})$ of the form

$$g^{ab}(\mathbf{q})=\Psi^{ab}g(\mathbf{q})\tag{69}$$

[where Ψ^{ab} satisfies the constraints (68)], the matrix equation (67) reduces to a scalar equation which in SCHA has a form

$$\lambda g(\mathbf{q})=\Lambda(\mathbf{q})g(\mathbf{q})-\sigma_1\int\frac{d^2\mathbf{q}'}{(2\pi)^2}g(\mathbf{q}'),\tag{70}$$

where

$$\Lambda(\mathbf{q})=G_0^{-2}(\mathbf{q}) \tag{71}$$

gives the spectrum of the replicon modes in absence of disorder ($\sigma_1\equiv 0$).

The lowest eigenvalue of Eq. (70) λ_0 should correspond to the real and rotationally symmetric eigenfunction $g_0(\mathbf{q})$ for which

$$\int \frac{d^2\mathbf{q}}{(2\pi)^2} g_0(\mathbf{q})\neq 0. \tag{72}$$

In that case the eigenfunction $g_0(\mathbf{q})$ can be excluded from Eq. (70) which can be rewritten as

$$\int \frac{d^2\mathbf{q}}{(2\pi)^2}\frac{\sigma_1}{-\lambda_0+\Lambda(\mathbf{q})}=1 \tag{73}$$

and therefore λ_0 can be negative only if the inequality

$$\int \frac{d^2\mathbf{q}}{(2\pi)^2}\frac{\sigma_1}{\Lambda(\mathbf{q})}>1 \tag{74}$$

is correct.

Since the expression in the lhs of inequality (74) is a product of an infinitely small factor σ_1 and a divergent factor

$$D=\int \frac{d^2\mathbf{q}}{(2\pi)^2}\frac{1}{\Lambda(\mathbf{q})}, \tag{75}$$

some regularization procedure has to be used to calculate it. Giamarchi and Le Doussal[7] have done it by adding to $G_0^{-1}(\mathbf{q})$ a small mass μ which in the end of calculation should be put down to zero. The other possible approach consists of restricting the integration in Eqs. (36) and (75) by the same constraint $q_m<|\mathbf{q}|<q_c$ with subsequent consideration of the limit $q_m\to 0$. With such a form of a regularization

$$\sigma_1=2Y(q_m/q_c)^{2K}, \tag{76}$$

whereas

$$D=\frac{1}{4\pi J^2}(q_m^{-2}-q_c^{-2}) \tag{77}$$

and therefore

$$\lim_{q_m\to 0}[\sigma_1(q_m)D(q_m)]=\begin{cases}\infty & \text{for } K<1,\\ \dfrac{Y}{2\pi J^2q_c^2} & \text{for } K=1,\\ 0 & \text{for } K>1.\end{cases} \tag{78}$$

Comparison with Eq. (74) allows then to conclude that for $K<1$ the lowest eigenvalue of Eq. (70) is negative and therefore the replica-symmetric solution is unstable.[7]

When deriving Eq. (73), which allows us to determine if the lowest eigenvalue of Eq. (67) is negative or not, only the first two terms of the expansion (62) for the Hessian were taken into account. An important sequence of higher-order corrections can be included into consideration if in Eq. (37) the bare value of fugacity is substituted by a scale-dependent fugacity $Y(q)$:

$$\sigma_1(q)=2Y(q)\exp[-G_0(R=0)]. \tag{79}$$

In terms of the Coulomb gas representation such substitution corresponds to a consistent addition to the contribution of a single charge contributions of the multicharge configurations which with the increase of scale become equivalent to a single charge. It seems reasonable to assume that the function $Y(q)$ should be given by the expression for $Y(R)$ in which the ratio of current and cutoff scales R/a is substituted by the inverted ratio of current and cutoff momenta:

$$Y(q)\equiv Y\left(R=a\frac{q_0}{q}\right). \tag{80}$$

Substitution of Eq. (79) [with $Y(q)$ defined by Eqs. (48), (49) and (80)] into the expression in the lhs of inequality (74) leads to an important change in the behavior for $K<1$ giving

$$\lim_{q_m\to 0}\int \frac{d^2\mathbf{q}}{(2\pi)^2}\frac{\sigma_1(q)}{\Lambda(\mathbf{q})}=\frac{1}{(aq_c)^2}t, \tag{81}$$

where

$$t=4K(1-K)\leqslant 1. \tag{82}$$

Comparison with criteria (74) shows that the inclusion of the renormalization effects removes the instability of the replica-symmetric state in the low-temperature phase. But this does not close the stability problem since it is necessary to consider another source for the instability which is maybe even more evident than the one discussed above, since it can be noticed even without any regularization. It can be associated with the neutral charge pairs which are always present in the system, that is one has to consider the second-order contribution to the Hessian (62) which remains finite even in the limit of infinite system size.

In the replica-symmetric case this contribution [the general form of which can be found by taking the second variation of Eq. (19)] acquires a form

$$L_2^{ab,cd}(\mathbf{q},\mathbf{q}')=-\frac{1}{2}P^{abcd}l_2(\mathbf{q},\mathbf{q}'), \tag{83}$$

which is characterized by the same dependence on replica indices as the first-order contribution (65), whereas its dependence on momenta is contained in the factor

$$l_2(\mathbf{q},\mathbf{q}')=Y^2\int d^2\mathbf{R}(2-2\cos\mathbf{qR})(2-2\cos\mathbf{qR}')W(R), \tag{84}$$

which can be also expressed in terms of the second-order contribution $\sigma_2(\mathbf{q})$ to the self-energy function

$$l_2(\mathbf{q},\mathbf{q}')=2\sigma_2(\mathbf{q})+2\sigma_2(\mathbf{q}')-\sigma_2(\mathbf{q}+\mathbf{q}')-\sigma_2(\mathbf{q}-\mathbf{q}'). \tag{85}$$

Since the dependence of $L_2^{ab,cd}(\mathbf{q},\mathbf{q}')$ on replica indices is given by the same matrix $\hat{P}$ as in the case of $L_1^{ab,cd}(\mathbf{q},\mathbf{q}')$, the eigenvalue equation for the replicon modes with the help of the same substituion (69) can be reduced to a scalar form:

$$\lambda g(\mathbf{q})=\Lambda(\mathbf{q})g(\mathbf{q})-\int \frac{d^2\mathbf{q}'}{(2\pi)^2}l_2(\mathbf{q},\mathbf{q}')g(\mathbf{q}'). \tag{86}$$

According to Eq. (52) in the high-temperature phase the behavior of $\sigma_2(\mathbf{q})$ at small $\mathbf{q}$ is given by

$$\sigma_2(\mathbf{q})\approx Bq^2 - Aq^{\nu}, \tag{87}$$

where

$$\nu=\begin{cases}4K-2 & \text{for } 1<K<3/2,\\ 4 & \text{for } 3/2<K.\end{cases}$$

Substitution of Eq. (87) into Eq. (85) shows that only the second term from Eq. (87) makes a contribution to $l_2(\mathbf{q},\mathbf{q}')$ whereas the contribution from the first term completely drops out. The simple power counting allows one then to conclude that in this case the zeroth-order term (71) dominates over the second-order term (85) (at least for small enough disorder). At $K=1$ the exponent ν becomes equal to 2 and approximation (87) ceases to be valid making the power counting arguments insufficient. For $K\leqslant 1$ a different approach should be used.

If $g(\mathbf{q})$ is the solution of Eq. (86) the corresponding eigenvalue λ is given by the functional

$$\Phi\{g\}=\frac{1}{I_N}(I_0-I_2), \tag{88}$$

where

$$I_N\equiv\langle g|g\rangle=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\,g(\mathbf{q})g(-\mathbf{q})\equiv\int d^2\mathbf{R}|g(\mathbf{R})|^2 \tag{89}$$

is the normalization integral, whereas

$$I_0\equiv\langle g|\hat{l}_0|g\rangle=J^2\int d^2\mathbf{R}(\nabla^2 g)^2 \tag{90}$$

and

$$I_2\equiv-\langle g|\hat{l}_2|g\rangle=\int d^2\mathbf{R}\;w(R)[g(0)-g(\mathbf{R})]^2 \tag{91}$$

can be interpreted as the matrix elements of the zeroth- and second-order contributions to Hessian. To simplify the discussion of the renormalization effects we have introduced in Eq. (91) the notation

$$w(R)=4Y^2W(R). \tag{92}$$

The lowest eigenvalue of Eq. (86) λ_0 corresponds to the eigenfunction $g_0(\mathbf{q})$ [or $g_0(\mathbf{R})$] which gives the absolute minimum of $\Phi\{g\}$. Since both expressions (90) and (91) are rotationally symmetric $g_0(\mathbf{R})$ also has to be rotationally symmetric:

$$g_0(\mathbf{R})\equiv g_0(R). \tag{93}$$

Substitution into Eq. (88) of any other function $g(\mathbf{R})$ can lead only to the increase of $\Phi\{g\}$:

$$\lambda_0<\Phi\{g(\mathbf{R})\}. \tag{94}$$

It is possible to show that for $K<1$ the second-order contribution to Hessian makes the replica-symmetric solution unstable by substituting into Eqs. (88)–(91) the arbitrary wave function with large enough localization radius. For example the substitution of the Gaussian wave function

$$g(R)=\frac{1}{\sqrt{\pi}r}\exp\left(-\frac{R^2}{2r^2}\right) \tag{95}$$

(in which the prefactor is specially chosen to make I_N equal to one) into Eqs. (90) and (91) for $K>1/2$ gives, respectively,

$$I_0=2\frac{J^2}{r^4} \tag{96}$$

and

$$I_2=4C(K)Y^2\left(\frac{a}{r}\right)^{4K}, \tag{97}$$

where

$$C(K)=(1-2^{2-2K})\Gamma(1-2K). \tag{98}$$

For $K\rightarrow 1$ the factor $C(K)$ has a finite limit:

$$\lim_{K\rightarrow 1}C(K)=\ln 2\approx 0.693, \tag{99}$$

whereas for $K\rightarrow 1/2$ the factor $C(K)$ diverges. Equations (96) and (97) can be expected to be valid only for $r\gg a$ when the details of the form of the cutoff are unimportant.

Comparison of Eq. (96) with Eq. (97) shows that for $K>1$ (and $r\gg a$) I_2 can be larger than I_0 only if Y is large enough. In contrast to that for $K<1$ for arbitrarily small Y one can make $\Phi\equiv I_0-I_2$ negative by chosing a sufficiently large localization radius r. The optimal value of localization radius r_0 for which the minimum of $\Phi(r)$ is achieved can be found by differentiating $\Phi(r)$ with respect to r. For $K\rightarrow 1-0$ and not too large Y this optimal radius diverges according to

$$\ln\frac{r_0}{a}\propto\frac{1}{1-K}, \tag{100}$$

and therefore for $K=1$ the stability of the replica-symmetric solution has to be determined by its large scale behavior.

But once again the conclusion about the instability of the replica-symmetric solution for $K<1$ holds true only if the renormalization effects are not taken into account. In the framework of a more general consideration it is possible to add to the contribution to the Hessian from the neutral pairs of charges [Eq. (83)], the contributions from the multicharge configurations which on large scales behave themselves in the same way as neutral pairs by substituting in Eq. (92) the bare fugacity by the scale dependent one. This changes the large scale behavior of the factor $w(R)$ into the form

$$w(R)=\left(\frac{1-K}{\pi}\right)^2\frac{1}{R^4}, \tag{101}$$

which in terms of the original problem corresponds to

$$K_{\rm eff}=1;\quad Y_{\rm eff}=\frac{1-K}{2\pi a^2}. \tag{102}$$

Equations (102) show that the renormalization effects shift the problem from the region of evident instability into the marginal situation $K_{\rm eff}=1$.

The direct substitution of the parameters given by Eq. (102) into Eq. (97) gives

$$\frac{I_2}{I_0}=\frac{\ln 2}{2}t^2\leqslant\frac{\ln 2}{2}<1, \tag{103}$$

which unfortunately does not provide any further insight since a different form of a trial wave function $g(R)$ may produce a larger ratio of I_2/I_0.

But still it turns out possible to prove that the sum of the operators $\hat{l}_0$ and $-\hat{l}_2^R$ is positively defined. By adding a superscript R we designate that the renormalization effects are assumed to be taken into account, that is the kernel $w(R)$ in the definition (91) of the operator $\hat{l}_2$ is chosen in the form (101). We are sticking to the asymptotic form of $w(R)$ since our previous estimate has shown that in the marginal case $K_{\rm eff}=1$ the localization radius r has to be infinite. Such analysis becomes even more reliable when the bare value of Y is smaller than $Y_{\rm eff}$. In that case according to Eqs. (48) and (49) the rhs of Eq. (101) gives an upper bound for its lhs and therefore the application of the asymptotic form (101) can only decrease the stability.

Since the eigenfunction $g_0(\mathbf{R})$ which corresponds to the global minimum of Φ has to be rotationally symmetric it is sufficient to discuss the form of this functional only for the rotationally symmetric functions $g(R)$. In that case Eq. (90) can be rewritten as

$$I_0=2\pi J^2\int_0^\infty dR\left[\frac{1}{R}\left(\frac{dg}{dR}\right)^2+R\left(\frac{d^2g}{dR^2}\right)^2\right]\equiv I_{01}+I_{02}, \tag{104}$$

where we have omitted the term

$$2\pi J^2\left[\left(\lim_{R\to 0}\frac{dg}{dR}\right)^2-\left(\lim_{R\to\infty}\frac{dg}{dR}\right)^2\right],$$

since both $\lim_{R\to 0}(dg/dR)$ and $\lim_{R\to\infty}(dg/dR)$ have to be equal to zero otherwise the integral in Eq. (104) would be divergent.

On the other hand, the expression (91) for I_2 after the substitution of the relation

$$g(0)-g(R)=-\int_0^R dR'\,\frac{dg(R')}{dR'} \tag{105}$$

with the help of inequality

$$2\frac{dg(R')}{dR'}\,\frac{dg(R'')}{dR''}\leqslant\left[\frac{dg(R')}{dR'}\right]^2+\left[\frac{dg(R'')}{dR''}\right]^2, \tag{106}$$

can be shown to satisfy

$$I_2\leqslant 2\pi\int_0^R dR\left[\frac{dg(R)}{dR}\right]^2\int_R^\infty dR'\,R'^2 w(R'). \tag{107}$$

For $w(R)$ of the form (101)

$$\int_R^\infty dR'\,R'^2 w(R')=\left(\frac{1-K}{\pi}\right)^2\frac{1}{R}. \tag{108}$$

Comparison of Eqs. (107) and (108) with Eq. (104) shows that the ratio of I_2 and the first term in Eq. (104) is never larger than one:

$$\frac{I_2}{I_{01}}\leqslant t^2\leqslant 1, \tag{109}$$

and therefore the operator $\hat{l}_0-\hat{l}_2^R$ is positively defined.

Thus we have shown that the inclusion of the renormalization corrections makes the second-order contribution to the Hessian not dangerous (earlier we have proved the same for the first-order contribution). But to consider both mechanisms simultaneously is a more difficult problem. Nonetheless the more attentive interpretation of the results obtained in this section allows us to conclude that we have proved that both $(aq_c)^{-2}t\hat{l}_0-\hat{l}_1^R$ and $t^2\hat{l}_0-\hat{l}_2^R$ are non-negative operators and therefore for

$$\frac{t}{(aq_c)^2}+t^2<1 \tag{110}$$

the operator $\hat{l}_0-\hat{l}_1^R-\hat{l}_2^R$ is positively defined. Even if our calculation has not been accurate enough to extract the correct numerical factors in front of t and t^2 in the inequality (110) (most probably they can also depend on the form of the cutoff), it still has to be valid for small enough t. That means that at least in some vicinity of the transition point $K_-<K<1$ (where $1/2<K_-<1$) the replica-symmetric solution has to remain stable.

Recently the analogous investigation of the stability of a replica-symmetric solution which takes into account the renormalization effects has been undertaken for the ϕ^4 problem with random field.[20,27,28]

V. THE SOLUTION WITH ONE-STEP REPLICA SYMMETRY BREAKING

A. General properties

In Refs. 6 and 7 the simplest nontrivial truncation (29) of the free-energy functional (16) has been considered which has been introduced as a result of the application of the variational approach. Remarkably the self-consistent equation for the correlation function which is obtained by a variation of Eq. (29) allows for the existence not only of the replica-symmetric solution but also of the solutions with the broken replica symmetry.

The renormalization-group approaches developed in Refs. 9–11 give no opportunities to discuss such solutions, since they explicitly assume that the correlation function (or the charge interaction) remains replica symmetric. It is maybe worthwhile to emphasize that in the renormalization-group description the Hamiltonian in the low-temperature phase remains essentially nonharmonic at arbitrarily large scales[10] and therefore the problem of finding the correct structure of the correlation function is in some sense postponed but never solved.

It has been shown that in the case of the two-dimensional system with Hamitonian (7) the simplest possible form of a replica symmetry breaking, namely, the one-step replica symmetry breaking is realized.[6,7] The case of the one-step replica symmetry breaking corresponds to such a form of a self-energy matrix $\Sigma^{ab}(\mathbf{q})$ when its nondiagonal elements can acquire only two different values {which it will be convenient to denote as $-[\sigma^{(0)}(\mathbf{q})+\sigma^{(1)}(\mathbf{q})]$ and $-\sigma^{(0)}(\mathbf{q})$} depending on whether the two indices a and b belong to the same block of the length m or not:

$$\Sigma^{ab}(\mathbf{q})=[n\sigma^{(0)}(\mathbf{q})+m\sigma^{(1)}(\mathbf{q})]\delta^{ab}-\sigma^{(1)}(\mathbf{q})\delta^{a'b'}-\sigma^{(0)}(\mathbf{q}). \tag{111}$$

The form of the first term in Eq. (111) follows from Eq. (26). Here and further on the indices with the prime denote the number of the block $(1\leqslant a',b'\leqslant n/m)$.

For $\Sigma^{ab}(\mathbf{q})$ of the form (111) inversion of Eq. (22) produces the expression which in the limit of $n\to 0$ reduces to

$$G^{ab}(\mathbf{q})=G_1(\mathbf{q})\delta^{ab}+\frac{1}{m}[G_0(\mathbf{q})-G_1(\mathbf{q})]\delta^{a'b'}+G_0^2(\mathbf{q})\sigma^{(0)}(\mathbf{q}), \tag{112}$$

where

$$G_1(\mathbf{q})=\frac{1}{G_0^{-1}(\mathbf{q})+m\sigma^{(1)}(\mathbf{q})}. \tag{113}$$

The obvious requirement for the size of the block to be between 1 and n in the limit of $n\to 0$ is transformed[4] into

$$0<m<1, \tag{114}$$

the limit of $m\to 1$ corresponding to the disappearance of replica symmetry breaking.

When Eq. (112) is substituted into any expression of the form (33) the last term (which is independent on replica indices) always drops out (like it does the analogous term in the replica-symmetric case). Therefore the disorder-induced contribution to free energy per replica $f_{\rm int}$ can be considered as a functional of $G_0(\mathbf{q})$ and $G_1(\mathbf{q})$. The matrix equation (22) can be then decoupled into two scalar equations:

$$\sigma^{(0)}(\mathbf{q})=-8\pi^2\frac{\delta f_{\rm int}}{\delta G_0(\mathbf{q})}, \tag{115}$$

$$\sigma^{(1)}(\mathbf{q})=-\frac{8\pi^2}{1-m}\frac{\delta f_{\rm int}}{\delta G_1(\mathbf{q})}, \tag{116}$$

the first of which has the same form as Eq. (34) for the replica-symmetric solution. Since $f_{\rm int}$ does not depend on $\sigma^{(0)}(\mathbf{q})$ we actually have to solve not the system of two equations but a single Eq. (116) whereas $\sigma^{(0)}(\mathbf{q})$ can be found by substituting the solution of Eq. (116) into Eq. (115).

Substitution of Eq. (112) into Eq. (21) shows that in the case of the one-step replica symmetry breaking the energy of a single charge can acquire two different values

$$E_0=G_1(\mathbf{R}=0)+\frac{1}{m}[G_0(\mathbf{R}=0)-G_1(\mathbf{R}=0)], \tag{117}$$

$$E_1=G_1(\mathbf{R}=0), \tag{118}$$

the first of which corresponds to the case when both indices α and β labeling the vector charge $\mathbf{s}$ belong to different blocks, whereas the second corresponds to the case when these indices belong to the same block. According to Eq. (113) $G_1(\mathbf{q})\leqslant G_0(\mathbf{q})$ and therefore for $0<m<1$,

$$E_0\geqslant G_0(\mathbf{R}=0). \tag{119}$$

B. Self-consistent harmonic approximation

In the SCHA only the lowest-order contribution to $f_{\rm int}$ has to be taken into account which in terms of E_0 and E_1 can be written as

$$f_1=Y[m\ \exp(-E_0)+(1-m)\exp(-E_1)]. \tag{120}$$

For $f_{\rm int}$ of the form (120) Eqs. (115) and (116) reduce to

$$\sigma_1^{(0)}(\mathbf{q})=2Y\exp(-E_0), \tag{121}$$

$$\sigma_1^{(1)}(\mathbf{q})=2Y\exp(-E_1)-2Y\exp(-E_0). \tag{122}$$

Since in two dimensions $G_0(\mathbf{R}=0)$ is logarithmically divergent we can conclude that in the framework of SCHA $\sigma^{(0)}(\mathbf{q})$ is always equal to zero, whereas according to Eq. (122) $\sigma^{(1)}(\mathbf{q})$ does not depend on $\mathbf{q}$. Thus instead of considering the free-energy functional which depends on two functions of $\mathbf{q}$ it is sufficient to consider the free energy which depends only on two variables $\Delta\equiv m\sigma_1^{(1)}/J$ and m:[6]

$$\begin{aligned}f&\equiv\lim_{n\to 0}\frac{1}{n}[F_{\rm VAR}(\Delta)-F_{\rm VAR}(\Delta=0)]\\&=\frac{J}{2}\left(\frac{1}{m}-1\right)\int_0^{\Delta}d\Delta'\Delta'\frac{dg(\Delta')}{d\Delta'}+Y(1-m)\exp[-g(\Delta)]\\&=\frac{1}{8\pi}\left(1-\frac{1}{m}\right)\Delta_c\ln\frac{\Delta_c+\Delta}{\Delta_c}+Y(1-m)\left(\frac{\Delta}{\Delta_c+\Delta}\right)^K.\end{aligned} \tag{123}$$

In the last line of Eq. (123) the function

$$g(\Delta)=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\frac{1}{J(q^2+\Delta)}, \tag{124}$$

which describes the fluctuations' width for the given value of the gap Δ, is assumed to be of the form corresponding to the sharp cutoff at $|\mathbf{q}|=q_c$:

$$g(\Delta)=K\ \ln\frac{\Delta_c+\Delta}{\Delta};\quad K=\frac{1}{4\pi J}, \tag{125}$$

where $\Delta_c=q_c^2$.

Variation of the free energy (123) with respect to Δ reproduces Eq. (122) which in terms of Δ and Δ_c can be rewritten as

$$\frac{\Delta}{m}=\frac{2Y}{J}\left(\frac{\Delta}{\Delta_c+\Delta}\right)^K. \tag{126}$$

If one looks for the maximum of the free-energy functional (as one is supposed to do in the replica representation) it is necessary also to take the variation of the free energy with respect to m, which gives

$$\frac{\Delta_c}{8\pi m^2}\ln\frac{\Delta_c+\Delta}{\Delta_c}=Y\left(\frac{\Delta}{\Delta_c+\Delta}\right)^K. \tag{127}$$

Thus we have obtained for two variables Δ and m a system of two equations from which they can be found. One can easily exclude from this system the variable m obtaining a single equation:

$$\left(\frac{\Delta}{\Delta_c}\right)^2=\gamma\ \ln\left(1+\frac{\Delta}{\Delta_c}\right)\left(\frac{\Delta}{\Delta_c+\Delta}\right)^K, \tag{128}$$

the properties of the solutions of which depend on the value of parameter

$$\gamma=\frac{Y}{2\pi J^2\Delta_c}. \tag{129}$$

For $J\propto T^{-1}$ and $Y\propto T^{-2}$ the parameter γ does not depend on temperature and therefore can be used to characterize the disorder strength.

For $\gamma<1$ the only solution of Eq. (128) for $K\geqslant 1$ is $\Delta=0$ but for $K<1$ the other solution also exists[6,7] which for $K\to 1-0$ is given by

$$\Delta\approx\Delta_c\gamma^{1/(1-K)}, \tag{130}$$

that is with increase in temperature the value of the gap Δ disappears in a continuous way, the singularity in the free energy being weaker than algebraic (''infinite order'' phase transition). For $\gamma\ll 1$ Eq. (130) is applicable not only for $1-K\ll 1$ but in the whole interval $0\leqslant K<1$.

For $\gamma>1$ the value of the gap Δ decreases with the increase in temperature not so rapidly and at $K=1$ remains finite and disappears (discontinuously) only at $K_c(\gamma)>1$. Since substitution of Eqs. (126) and (127) into Eq. (123) gives

$$f=\frac{J(1-m)^2\Delta}{2m}, \tag{131}$$

and on the phase-transition line the values of the free energies of two phases (one with $\Delta>0$ and another with $\Delta=0$) have to coincide, this can happen only when $m=1$. This observation allows us to provide, with the help of Eqs. (126) and (127), a parametrical description of phase-transition line for $\gamma>1$:

$$K=\frac{x}{\ln(1+x)}, \tag{132}$$

$$\gamma=\frac{x^2}{\ln(1+x)}\left(\frac{1+x}{x}\right)^{x/[\ln(1+x)]}, \tag{133}$$

the value of the gap at the transition being given by $\Delta=x\Delta_c$.

The form of Eq. (131) suggests that although for $\gamma>1$ the gap disappears discontinuously the form of the singularity in the free energy corresponds to a second-order phase transition (the jump in the heat capacity). This is rather natural since at $m=1$ one cannot distinguish the phase with the broken replica symmetry from the phase without it and therefore the transition cannot be of the first order. Note that the position of the tricritical point ($\gamma=1$) coincides with the point at which the replica-symmetric solution at $K=1$ becomes (in the framework of SCHA) unstable [Eq. (78)]. However it cannot be excluded that the change in the transition type with the increase of disorder can be an artefact of SCHA.

Substitution of Eq. (112) into Eq. (56) shows that for the solution with the one-step replica symmetry breaking the irreducible part of the correlation function also remains unrenormalized like in the replica-symmetric case (this is a general property not related to SCHA). On the other hand, according to Eq. (58) the reducible part becomes nonzero already in the lowest-order approximation as soon as $\sigma^{(1)}\times(\mathbf{q})$ is finite. In SCHA $\sigma^{(0)}(\mathbf{q})$ is always equal to zero and therefore only the second term from Eq. (112) makes a contribution to $C_r(\mathbf{R})$:

$$C_r(\mathbf{R})=\frac{1-m}{m}\int\frac{d^2\mathbf{q}}{(2\pi)^2}(2-2\ \cos\mathbf{qR})[G_0(\mathbf{q})-G_1(\mathbf{q})]$$
$$\approx\frac{1-m}{m}\times\begin{cases}K\left(\dfrac{R}{\xi}\right)^2\ln\dfrac{\xi}{R} & \text{for } a\ll R\ll\xi,\\[2ex] 4K\ \ln\dfrac{R}{\xi} & \text{for } \xi\ll R,\end{cases} \tag{134}$$

where the correlation length ξ is defined by the relation $\Delta=\xi^{-2}$. Summation of Eqs. (57) and (134) shows that the full correlation function $C(R)$ behaves as

$$C(R)\approx\begin{cases}4K\ \ln(R/a) & \text{for } a\ll R\ll\xi,\\[1ex] 4K\ \ln(\xi/a)+\dfrac{4K}{m}\ln(R/\xi) & \text{for } \xi\ll R,\end{cases} \tag{135}$$

that is, at R of the order of ξ a continuous crossover has to take place between the two different values of the prelogarithmical factor. For the small scales the prelogarithmical factor should be just the same as in absence of disorder.[7]

Substitution of Eq. (127) into Eq. (126) allows us to obtain a relation

$$\frac{K}{m}=\frac{\Delta/\Delta_c}{\ln(1+\Delta/\Delta_c)}, \tag{136}$$

which shows that as soon as $\Delta/\Delta_c>0$ the ratio K/m is larger than one (and increases with the increase of Δ that is with the decrease of temperature). Therefore the asymptotic value of the prelogarithmical factor at the transition line has a minimum and discontinuous derivative.[6]

In terms of the Coulomb gas representation the appearance of the gap with the one-step replica symmetry breaking structure corresponds to debounding of some types of pairs whereas the pairs of the other types remain bound. On the total there are $n(n-1)/2$ types of noncollinear elementary vector charges (which can be numbered by two indices $1\leqslant\alpha<\beta\leqslant n$). According to the form of Eq. (113) for $\sigma^{(1)}(\mathbf{q}=0)>0$ the energy E_1 of the charges for which both indices belong to the same block becomes finite whereas the energy E_0 of the charges with the indices from the different blocks remains logarithmically divergent. Simple comparison shows that the parameter m gives the fraction of the total number of the types of charges which remain bound at the given temperature. With a decrease of temperature down to $K=0$ this fraction goes to zero. Such interpretation of m is completely compatible with the requirement (114).

C. Stability analysis

To show that the solution with the replica symmetry broken by one step can be really discussed as a possible candidate for the description of the properties of the system one has to check the stability of this solution. We shall do it in the framework of a more general approach when the size of the block m is not fixed by the requirement $\delta f/\delta m=0$ but is kept as a free parameter. Strictly speaking for any m a solution of Eqs. (115) and (116) defines a solution of the matrix equation (22) and therefore can be discussed in the same fashion as the extremal solution.

In the case of the one-step replica symmetry breaking the zeroth-order contribution to the Hessian retains its form (63) [the difference with the replica-symmetric case being that now the form of $[G^{-1}(\mathbf{q})]^{ab}$ should correspond to $\Sigma^{ab}(\mathbf{q})$ of the form (111)] whereas the first-order contribution can be written as

$$L_1^{ab,cd}(\mathbf{q},\mathbf{q}')=-\frac{1}{2}P^{abcd}\sigma_1^{(0)}-\frac{1}{2}P_1^{abcd}[\sigma_1^{(1)}-\sigma_1^{(0)}], \tag{137}$$

where the matrix $\hat{P}_1$ is defined by the equation

$$P_1^{abcd}=\frac{1}{2}\sum_{\mathbf{s}}{}' s^a s^b s^c s^d, \tag{138}$$

which is of the same form as Eq. (66) with the exception that in Eq. (138) the summation should be restricted only to those vector charges $\mathbf{s}$ for which both indices α and β belong to the same block. We are keeping $\sigma_1^{(0)}$ in Eq. (137), although according to Eq. (119) in infinite two-dimensional systems $\sigma_1^{(0)}\to 0$, since like in the replica-symmetric case when the stability is concerned even such vanishing quantities can be of importance.

In the case of one-step replica symmetry breaking the two families of eigenfunctions of the general equation (67) are potentially dangerous. For the first of them the dependence of $g^{ab}(\mathbf{q})=\Psi^{ab}g(\mathbf{q})$ on replica indices a and b is described by the matrix $\Psi^{ab}\equiv\Psi^{a'b'}$, the elements of which are all equal to each other inside each of $n/m\times n/m$ blocks of size $m\times m$ but can be different in different blocks, satisfying nontheless the whole set of constraints (68). The eigenstate belonging to this family can be described as a block replicon since it has the same structure as replicon but is constructed from the uniform blocks instead of separate elements. In particular, the diagonal blocks can contain only zero elements.

The matrix equation (67) for the block replicons reduces to scalar equation which in SCHA differs from Eq. (70) for the replica-symmetric case only by substitution:

$$\sigma_1\Rightarrow m^2\sigma_1^{(0)}.$$

Therefore the same analysis as in Sec. IV can be applied the only difference being that the expression for σ_1 given by Eq. (76) should be substituted by

$$m^2\sigma_1^{(0)}=2Ym^2\left(\frac{\Delta}{\Delta_c+\Delta}\right)^{-[(1-m)/m]K}\left(\frac{q_m}{q_c}\right)^{2K/m}. \tag{139}$$

Comparison with Eq. (77) shows then that the lowest eigenvalue among the eigenstates belonging to the class of block replicons remains non-negative for

$$m<K\equiv m_{\max}(K), \tag{140}$$

whereas for $m=m_{\max}(K)$ the situation is marginal and the answer depends on the relation between parameters.

The other class of eigenfunctions which are potentially dangerous can be called the in-block replicons since each of them can be described by the matrix Ψ^{ab} the elements of which are nonzero only inside of one of n/m diagonal blocks of size $m\times m$ but satisfy all the constraints (68). For this family of eigenfunctions the matrix equation (67) also reduces to a scalar equation which in SCHA has a form

$$\lambda g(\mathbf{q})=\Lambda_1(\mathbf{q})g(\mathbf{q})-\sigma_1^{(1)}\int\frac{d^2\mathbf{q}}{(2\pi)^2}g(\mathbf{q}), \tag{141}$$

where

$$\Lambda_1(\mathbf{q})=G_1^{-2}(\mathbf{q})=J^2(q^2+\Delta)^2. \tag{142}$$

Thus we have again obtained the equation with the same structure as Eq. (70) but now there is no need to introduce any regularization since for the finite Δ both

$$\sigma_1^{(1)}=2Y\left(\frac{\Delta}{\Delta_c+\Delta}\right)^K \tag{143}$$

and

$$D=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\frac{1}{\Lambda_1(\mathbf{q})}=\frac{1}{4\pi J^2}\frac{\Delta_c}{\Delta(\Delta_c+\Delta)} \tag{144}$$

are finite.

According to the analysis of Sec. IV the border between the stable and unstable region corresponds to the case when the product of $\sigma_1^{(1)}$ and D is equal to one:

$$\gamma\frac{\Delta_c}{\Delta}\left(\frac{\Delta_c}{\Delta_c+\Delta}\right)^{K+1}=1. \tag{145}$$

Although this equation does not contain m explicitly it contains $\Delta\equiv\Delta(m,K)$ which should be chosen as a solution of Eq. (126) for the given values of m and K. With the help of Eq. (126), Eq. (145) can be transformed into the equation for $m_{\min}$ which defines the lower border of the stability interval

$$m_{\min}(K)<m<m_{\max}(K) \tag{146}$$

and is of the following form:

$$m_{\min}=\frac{\Delta_c}{\Delta_c+\Delta(m_{\min},K)}K. \tag{147}$$

Comparison of Eqs. (136), (140), and (147) allows us to conclude that the inequality

$$m_{\min}(K)<m_0(K)<m_{\max}(K), \tag{148}$$

where $m_0(K)$ is the value of m for which the free energy $f(m)\equiv f[\Delta(m,K),m]$ has a maximum is always correct.

That means that the replica symmetry breaking solution corresponding to the maximum of the free energy (the extremal solution) is always stable.

D. Unimportance of the higher-order corrections to the replica symmetry breaking solution

As has been discussed above the difference between the replica-symmetric and replica symmetry breaking solutions manifests itself in the behavior of the reducible part of the correlation function. For the replica symmetry breaking solution it diverges in the same way as the irreducible part (that is logarithmically) whereas for the replica-symmetric solution the divergence is faster (the square of logarithm). But the properties of two solutions have been found with the help of different approximations. The form of the replica symmetry breaking solution has been derived with the help of SCHA which corresponds to keeping in the Dyson equation only the lowest-order term in the expansion for the self-energy matrix $\Sigma^{ab}(\mathbf{q})$. On the other hand in the case of the replica-symmetric solution, not only the second-order term but also the important sequence of higher-order terms of this expansion are taken into account, all of which are completely neglected in SCHA. Thus it is definitely necessary to check if the slower divergence of the correlation function for the replica symmetry breaking solution is really an intrinsic property of this solution and cannot be explained by the insufficient accuracy of SCHA which neglects exactly those terms in the expansion for $\Sigma^{ab}(\mathbf{q})$ which are responsible for the faster divergence of the correlation function of replica-symmetric solution.

In the replica-symmetric case all the peculiarities in the behavior of the correlation function can be related only to the last term of Eq. (32) which does not depend on replica indices but is proportional to the self-energy function $\sigma(\mathbf{q})$. An analogous term is present also in Eq. (112) which defines the general form of $G^{ab}(\mathbf{q})$ for the case of one-step replica symmetry breaking, but in SCHA it does not play any role since the lowest-order contribution to it [given by Eq. (121)] in the thermodynamic limit is always equal to zero (like in the replica-symmetric case).

The first nonvanishing contribution to $\sigma^{(0)}(\mathbf{q})$ appears in the second-order in Y (also like in the replica-symmetric case). Since $\sigma^{(0)}(\mathbf{q})$ is defined by the relation (115), and the energy (118) of the free (unbound) charges as well as their interaction with the other charges depend only on $G_1(\mathbf{q})$ [but not on $G_0(\mathbf{q})$], only the bound pairs of charges make a finite contribution to $\sigma_2^{(0)}(\mathbf{q})$.

In the replica-symmetric case the total energy of two vector charges is finite only for the neutral pair. When the replica symmetry breaking gap appears, that is no longer so. For the energy of the pair to be finite the charges $\mathbf{s}_i$ and $\mathbf{s}_j$ have only to belong to the opposite nondiagonal blocks ($\alpha_i'=\beta_j'$, $\beta_i'=\alpha_j'$), that is the pair has to be neutral only with respect to the index numbering the blocks. The energy of such a pair

$$E_p(R)=2[G_2(0)-G_2(R)]+2G_1(0)+(\mathbf{s}_i\mathbf{s}_j)G_1(R) \tag{149}$$

can acquire three different values depending on the product $(\mathbf{s}_i\mathbf{s}_j)$ (which for the pair with the finite energy can be equal to 0, -1 or -2). To simplify the equations in this subsection we have introduced in Eq. (149) a notation

$$G_2(R)=\frac{1}{m}[G_0(R)-G_1(R)]. \tag{150}$$

The sum of the contributions from all three types of bound pairs of charges can be written in the form

$$\sigma_2^{(0)}(\mathbf{q})=Y^2\int d^2\mathbf{R}(2-2\cos\mathbf{qR})\exp\{-2[G_2(0)-G_2(R)]-2G_1(0)\}[\exp G_1(R)+m-1]^2, \tag{151}$$

which in the limit of $m\to1$ reduces to Eq. (38) of the replica-symmetric case.

The contribution to Eq. (149) related to $G_1(R)$ is of a minor importance since it remains finite for $R\to\infty$ and the logarithmical divergence of $E_p(R)$ at large scales is related entirely to $G_2(R)$:

$$E_p(R)\approx2[G_2(R=0)-G_2(R)]\approx\frac{4K}{m}\ln(R/a). \tag{152}$$

Comparison with Eq. (40) shows that although the appearance of the replica symmetry breaking gap leads to debounding of some charges the interaction of the charges in the remaining bound pairs becomes even stronger. In Sec. IV B it has been shown that the ratio K/m is always larger than one and therefore the expression (151) for small q can be always approximated as

$$\sigma_2^{(0)}\approx Bq^2, \tag{153}$$

where B is given by a convergent integral like in the high-temperature phase.

Note that as in the replica-symmetric case there is no feedback in calculation of the replica-symmetric contribution to the self-energy part $\sigma^{(0)}(\mathbf{q})$. In the limit of $n\to0$ $\sigma^{(0)}(\mathbf{q})$ drops out from any expression for the charge-charge interaction [of the form (20)] and therefore is not present in the rhs of Eq. (151). Thus we have shown that in the case of the replica symmetry breaking solution the logarithmically divergent contribution to the correlation function related with the presence of bound pairs also exists, but in contrast to the replica-symmetric solution the prelogarithmical coefficient is given by a convergent expression.

The model (7) which we consider here has one very important advantage with respect to some other problems with similar Hamiltonian (for example the random manifold problem[25]) that it can be described in terms of a Coulomb gas. And in a Coulomb gas description any divergence can be associated only with the form of the interaction between the charges or some particular complexes of charges. This allows us to check for the appearance of the new divergencies in the higher orders of the expansion simply by checking if combining some charges together one can construct the objects with the lower interaction between them than the interaction between the elementary charges.

Quite often this is not possible and that is why in some of the Coulomb gas problems the exact form of critical behavior can be found while keeping only a finite number of terms in the renormalization-group equations. This applies, for example, to the ordinary (scalar) Coulomb gas which is isomorphic to the one-component sine-Gordon model[23] and also to the variety of the vector Coulomb gases[29–32] including the replica-symmetric Coulomb gas[10] which describes the properties of the replica-symmetric solution of our problem. Therefore the results reviewed in Sec. III can be expected to give a quantitatively correct description of the replica-symmetric solution of the Dyson equation even in the critical region.

In the case of the replica symmetry breaking solution the situation is more complex than in replica-symmetric case since instead of constructing the renormalization procedure simply for the summation of all the essential contributions to the expression for the self-energy function it is necessary to solve simultaneously the equation for the replica symmetry breaking contribution to the self-energy part. But still the application of the Coulomb gas representation allows to understand that when the second-order contribution to B is convergent then all the higher-order contributions also do not contain any divergencies (like in the high-temperature phase). The only source for the appearance of divergences is related to the interaction between the bound charges and the same interaction appears in all the orders of the expansion in fugacity (i.e., in the number of the charges involved).

Thus we have shown that when the second- and higher-order terms are included into the calculation of the reducible part of the correlation function for the replica symmetry breaking solution it still diverges logarithmically, that is in the same fashion as the irreducible part. Therefore the full correlation function (5) is also characterized by the logarithmical behavior, but with a larger prefactor than predicted by SCHA (Refs. 6 and 7) which takes into account only the contribution from the unbound charges.

Although the inclusion of higher-order corrections turned out to be of no importance for the form of long-distance behavior of the correlation function $C(R)$ it may lead to some quantitative changes. Unfortunately it is not easy to check how strong they are since, if the second-order corrections are taken into account $\sigma^{(1)}(\mathbf{q})$ cannot remain independent of the momenta. The only thing we have been able to check is that at low enough temperatures the second-order correction to the free energy becomes much smaller than the first-order contribution.

For $\gamma \ll 1$ and $K \ll 1/\ln(1/\gamma)$ when Eq. (130) reduces to $\Delta \sim \gamma \Delta_c$ the second-order term in the expression for the free energy per replica

$$f_2 = \lim_{n\to 0} \frac{1}{n} F_2 \tag{154}$$

is dominated by the contribution from the bound pairs of charges and after substitution of $G^{ab}(\mathbf{q})$ of the form (112) can be estimated as

$$f_2 \sim \frac{Y^2 K^3}{\Delta}, \tag{155}$$

where in accordance with $\gamma \ll 1$ we have used the relation $m \approx K$. Comparison with $f_1 \sim Y$ shows that for such temperatures $f_2/f_1 \sim K \ll 1$ which gives some hope that the higher-order corrections may be even smaller.

With increase of temperature the higher order corrections become more important and can lead to the change in the magnitude of the gap with respect to the prediction of SCHA. To take this effect into account in a systematic way it is necessary to construct some renormalization procedure to describe the form of replica symmetry breaking solution. Recently the attempts to describe the replica symmetry breaking in the model (7) with the help of the renormalization-group formalism have been undertaken by Le Doussal and Giamarchi[33] and by Kierfeld.[34] But in these works a special situation was considered when the replica symmetry breaking term is artificially added to the Hamiltonian (7). Apparently such a consideration does not allow one to make any conclusions about the spontaneous replica symmetry breaking when not the Hamiltonian but the correlation function (the solution of the Dyson equation) loses the replica symmetry. Therefore the problem of the renormalization-group description of the replica symmetry breaking solution still remains to be solved.

VI. DISCUSSION

In the present work we have investigated the properties of the different solutions of the Dyson equation which appears in the replica approach to the simplest model of a two-dimensional uniaxial vortex glass. In particular we have shown that the solution with the one-step replica symmetry breaking when it exists is always stable. On the other hand, the replica-symmetric solution proves to be stable not only in the high-temperature phase but also in the low-temperature phase (at least in some finite temperature interval). Thus the simplest possibility, when in the low-temperature phase (in which two solutions coexist) one is deprived of the necessity to make the choice between them since one of these solutions is unstable, is not realized.

Therefore the more general principles should be applied and the established point of view is that in the situation with the replica symmetry breaking one always has to look for the solution with the maximal free energy.[4] In the domain where the solution with the one-step replica symmetry breaking exists it always has the larger free energy and since it is always stable such choice is not in contradiction with the stability requirements.

The coexistence of two stable solutions one of which is replica-symmetric whereas the other corresponds to the one-step replica symmetry breaking is known to occur also in so-called p-spin spherical model of a spin glass with infinite interaction range.[35] In this system the choice of the replica symmetry breaking solution is supported also by the results of the self-consistent dynamic approach[36] which shows that the system of equations for the response and correlation function in the domain of the phase diagram where the replica symmetry breaking can occur does not allow for an ergodic solution (which can be associated with the replica-symmetric solution of the replica representation).

The application of the same approach to the model (1) also predicts that in the low-temperature phase the ergodic solution of the self-consistent dynamic equations does not exist.[15] This gives an additional support to the choice of the replica symmetry breaking solution for the description of the system in replica representation. But this support is probably not so strong as in the case of the p-spin spherical spin-glass model, since in the spin-glass model with infinite interaction range the equations of self-consistent dynamics are supposed to give the exact description of the system whereas in the case of the model (1) they correspond to keeping only the lowest-order nontrivial terms in the complete equations for the response and correlation functions. Since SCHA can be described in exactly the same terms (it also corresponds to keeping only the lowest-order nontrivial term in the self-consistent equation—but in replica representation) the predictions and limitations of both these approaches can be expected to be in some correspondence with each other.

In Sec. IV we have shown that the prediction of SCHA for the instability of the replica-symmetric solution is invalidated if the renormalization of the fugacity is taken into account. Therefore it may be important to check if the same does not happen with the prediction of the self-consistent dynamic approach for the dynamic instability of the ergodic solution at low temperatures.[15]

The higher-order corrections to the dynamic correlation function have been considered by Goldshmidt and Schaub[14] and by Tsai and Shapir.[16] These authors have developed in the framework of the dynamic description the renormalization scheme which produces for the static correlation function the same renormalization-group equations as have been earlier found in the replica approach for the description of the replica-symmetric solution.[10,11] Unfortunately in Refs. 14,16 the second-order contribution to the renormalization has been found only for the renormalization of the persistent part of the time-dependent correlation function (which allows one to establish the agreement with the results of the replica approach), whereas in the renormalization of the response function only the first-order contribution has been considered. Therefore such an approach may turn out to be of the same leval of reliabilty as the self-consistent dynamic approach[15] since both of them take into account only the lowest order nontrivial contributions to the dynamic equations. Moreover the analysis of Refs. 14 and 16 is performed in terms of frequency-dependent response and correlation functions which makes it rather complicated just to check if their time dependence is compatible with what should be expected for the ergodic solutions of the equations for purely relaxational dynamics (in the same fashion as is done in the self-consistent dynamic approach[15,37]).

It should be noted that although the form of the static correlation function in the low-temperature phase predicted by the nonergodic solution of the self-consistent dynamic equations is the same as predicted by the replica symmetry breaking solution of the replica representation, the parameters Δ and m which enter it correspond not to the extremal replica symmetry breaking solution [for which Δ and m are solutions of Eqs. (126) and (127)] but to the marginal replica symmetry breaking solution which corresponds to the lower border (147) of the stability interval for m.

This property is known to be a common feature of all the models in which the one-step replica symmetry breaking takes place. The list of examples includes in particular p-spin Ising[38,39] and p-spin spherical[35,36] versions of the infinite-range spin glass. One of the consequences of such a descrepancy is that in the case when the transition is discontinuous (in terms of the gap Δ) the dynamic approach predicts the phase transition at a higher temperature that follows from the consideration of the extremal replica symmetry breaking solution. In the model (1) considered in this work this happens for $\gamma>1$ whereas for $\gamma\leqslant 1$ the gap appears in a continuous way and the predictions of both approaches for the position of the phase-transition line coincide with each other. The reasons for the existence of such discrepancy between the predictions of the two methods (which for the infinite range systems both are supposed to provide the exact description) are not very well understood.

Recently a suggestion has been put forward[40] that in the case of one-step replica symmetry breaking one should look not for the maximum but for the minimum of the free energy with respect to the size of the block (among the stable solutions). Such an assumption allows one to eliminate the discrepancy with the predictions of the self-consistent dynamic approach but only in the narrow part of phase diagram in which the free energy of the marginal solution corresponding to $m_{\min}(K)$ is lower than the free energy of the replica-symmetric solution. We have checked numerically for the model (7) that in the domain $K<1$ where the upper bound (140) for m is not meaningless the free energy given by SCHA is always lower for $m=m_{\max}(K)$ than for $m=m_{\min}(K)$ and so the principle that the dynamic equations choose the solution with the lowest free energy among the stable solutions does not seem to work.

Since we have shown that the higher-order corrections do not change the prediction of SCHA for the logarithmical divergence of the correlation function in the low-temperature phase, the numerical simulations may be helpful to distinguish between the replica symmetric and replica symmetry breaking solutions. The simulations of the random phase discrete Gaussian model[18,41] and of the random phase sine-Gordon model[42] [both of which can be expected to demonstrate the same properties as the model (1)] have confirmed that in the low-temperature phase the slope of the curve $C(R)$ versus $\ln R$ increases with increase of R. Although the authors of Refs. 18, 41, and 42 make the suggestions that the observed behavior is compatible with one or another theoretical prediction, the absence of the agreement between the interpretations testifies rather that the additional simulations may be needed to resolve the difference between the logarithm squared and the logarithm with increasing slope.

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